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THE EXPLICIT SOLUTION OF THE EQUATIONS OF THE ELASTIC DEFORMATIONS FOR A STRATIFIED ROAD UNDER GIVEN STRESSES IN THE DYNAMIC CASE

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SYNOPSIS

A road is considered as an elastic stratified body in a three dimensional space. It is assumed that each layer is a homogenous material, characterized by its Lamé elastic constants (or its Young modulus and Poisson ratio) and its density. The system of partial differential equations which determine in the i-th layer and in function of the time the displacement components is solved under the following conditions: For t = 0, displacement components and their partial derivatives in t vanish; for t > 0, stress components are known functions on the free surface and, at the points of contact of two layers, the displacements and stresses are the same if computed in the upper or lower layer.

First, we solve the problem of finding the solution of the system in the i-th layer, knowing displacement or stress components at the upper or lower surfaces. For this, the notion of function is generalized using the Ehresmann local structures and the differential system is considered as a system of equations for the new objects; by an integral transformation the system is reduced to a system of linear equations. By solving this system, we obtain relations between the integral transforms of the displacement and stress components at the upper and lower intersurfaces of the i-th layer as well as at the upper surface of the last layer.

Assuming the displacement components known on the free surface, the integral transforms of the stress and displacement components are computed on the second intersurface as functions of these parameters. These quantities are also computed as functions of the displacement components at the last intersurface. By writing that the solution is the same in both cases, we are led to a system of linear equations, the solutions of which are the integral transforms of the displacement components at the free surface and at the last intersurface. From these, we deduce the solutions of the initial equations.

The solutions are obtained in the form of ordinary integrals containing the given stress components. These integrals can be computed with a computing machine or approximated by elementary functions (the approximation depends on respective sizes of the parameters).

The regularity conditions imposed for the given stresses are practically not restrictive; in particular, strain and stress components are not supposed to be harmonic.

Examples: stresses produced by a vibrating machine or by the movement of a vehicle.

INTRODUCTION

A road will be considered as an elastic stratified body in three dimensional space R^3. Let $(0, e_1, e_2, e_3)$ be an orthogonal frame of R^3, the coordinates of a point x being (x_1, x_2, x_3). It is assumed that the vector $0e_3$ is vertical and downward oriented and that the free surface of the road the surface $x_3 = 0$.

There are q layers of homogenous and isotropic materials, characterized for the p-th layer by Lamé's elastic constants λ^p and μ^p and density ρ^p; the constants λ^p and μ^p are obtained from the Young's modulus E^p and the Poisson's ratio σ^p by the formulas:

$$\lambda^p = \frac{\sigma^p E^p}{(1+\sigma^p)(1-2\sigma^p)} \; ; \; \mu^p = \frac{E^p}{2(1+\sigma^p)}$$

The p-th layer is defined by the inequalities: $h_{p-1} \leqslant x_3 < h_p$ for any $p < q$, and the q-th layer is defined by $h_{q-1} < x_3$.

The time parameter will be denoted by t.

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In the p-th layer, the displacement u^p has components (u_1^p, u_2^p, u_3^p) which are functions of x and t and solutions of the system (1):

$$\rho^p \frac{\partial^2 u_j^p}{\partial t^2} - \mu^p \Delta u_j^p - (\lambda^p + \mu^p) \frac{\partial \theta_u^p}{\partial x_j} = 0 \qquad (1)$$

where: $j = 1,2,3$

$$\Delta u_j^p = \frac{\partial^2 u_j^p}{\partial x_1^2} + \frac{\partial^2 u_j^p}{\partial x_2^2} + \frac{\partial^2 u_j^p}{\partial x_3^2} \qquad (2)$$

$$\theta_u^p = \frac{\partial u_1^p}{\partial x_1} + \frac{\partial u_2^p}{\partial x_2} + \frac{\partial u_3^p}{\partial x_3} \qquad (3)$$

Furthermore, the following boundary conditions are imposed:

1) For $t = 0$, the displacement components and their first partial derivatives in t vanish in every point.

2) For $t > 0$, the stress components on the free surface $x_3 = 0$ are given functions π_1, π_2, π_3 of (x_1, x_2, t) and we have the following equations induced on $x_3 = 0$:

$$\mu^1 \left(\frac{\partial u_3^1}{\partial x_1} + \frac{\partial u_1^1}{\partial x_3} \right) = \pi_1 \qquad (4)$$

$$\mu^1 \left(\frac{\partial u_3^1}{\partial x_2} + \frac{\partial u_2^1}{\partial x_3} \right) = \pi_2 \qquad (5)$$

$$2\mu^1 \frac{\partial u_3^1}{\partial x_3} + \lambda^1 \theta_u^1 = \pi_3 \qquad (6)$$

3) For $t > 0$, the displacement and stress components on the intersurface $x_3 = h_p$ are the same if computed in the p-th layer or in the $(p + 1)$-th layer, which leads to the following equations induced on $x_3 = h_p$:

$$u_j^{p+1} = u_j^p \quad , \quad \text{where } j = \begin{cases} 1 \\ 2 \\ 3 \end{cases} \qquad (7)$$

$$\mu^{p+1} \frac{\partial u_k^{p+1}}{\partial x_3} = \mu^p \frac{\partial u_k^p}{\partial x_3} + (\mu^p - \mu^{p+1}) \frac{\partial u_3^p}{\partial x_k}, \quad k = \begin{cases} 1 \\ 2 \end{cases} \qquad (8)$$

$$(\lambda^{p+1} + 2\mu^{p+1}) \frac{\partial u_3^{p+1}}{\partial x_3} = (\lambda^p + 2\mu^p) \frac{\partial u_3^p}{\partial x_3} + (\lambda^p - \lambda^{p+1}) \left(\frac{\partial u_1^p}{\partial x_1} + \frac{\partial u_2^p}{\partial x_2} \right) \qquad (9)$$

Thus the displacement is solution of a mixed Cauchy problem for the system (1). This problem is solved by the method sketched in (b), which is an application of the notion of distructures introduced in (c).

DISTRIBUTIONS

Let R^n be the n-dimensional space; a point x of R^n has coordinates $x_1, \ldots x_n$. An open set of R^n is a set Ω which contains, with each point x of Ω, an open disc $\Sigma_{j=1}^n (y_j - x_j)^2 < \epsilon$. A bounded set is a set which is contained in a disc.

In all this section, ϕ will denote an infinitely differentiable real-valued function defined in an open set Ω of R^n.

Let f be a continuous real-valued function defined in Ω; for every ϕ vanishing outside a bounded set contained in Ω, the (Lebesgue) integral: $f(\phi) = \int_\Omega f(x)\, \phi(x) dx$ is defined and: $\phi \to f(\phi)$ is a linear functional of ϕ denoted also by f.

For every set of integers $(i_1, \ldots, i_p)$, $\frac{\partial^p \phi}{\partial x_{i_1} \ldots \partial x_{i_p}}$ will be denoted by $\partial_{i_1 \ldots i_p} \phi$ and the mapping:

$$\phi \longrightarrow (-1)^p f(\partial_{i_1 \ldots i_p} \phi)$$

is a linear functional of ϕ denoted by $\partial_{i_1 \ldots i_p} f$ and called the functional $(i_1 \ldots i_p)$-partial derivative of f.

If f is p times continuously differentiable, this functional is the functional corresponding as above to the ordinary $(i_1 \ldots i_p)$-partial derivative $\partial_{i_1 \ldots i_p} f$.

Let Φ be the set of all the functionals defined above corresponding to all open sets Ω of R^n; an element F of Φ can be obtained in different ways, e.g., $F = \partial_{i_1 \ldots i_p} f = \partial_{j_1 \ldots j_q} g$. The domain in which F is defined will be denoted by $\Omega(F)$.

We may define an order in Φ in the following way: F is a restriction of G, or $F < G$, if and only if, for every ϕ vanishing outside a bounded set contained in $\Omega(F)$, we have: $F(\phi) = G(\phi)$.

With this order, Φ is a species of local structures spread over the set of the open sets of R^n, according to Ehresmann's definition. A distribution is an element of the complete species of local structures (d) associated with Φ: Two elements F and F' of Φ are compatible if, for every ϕ vanishing outside a bounded set contained in $\Omega(F)$ and in $\Omega(F')$, we have $F(\phi) = F'(\phi)$. A distribution is a complete family C of elements of Φ, that is a family satisfying the following conditions:

1) Two elements of C are compatible.
2) C contains, with F, all the restrictions of F.
3) Let $(F_i)_{i \in I}$ be a sub-family of elements of C such that $F_i < G$ for any i in I; then C contains the restriction of G to the set of all the points belonging at least to one $\Omega(F_i)$.

The set $\Omega(C)$ of all the points belonging at least to one $\Omega(F)$, where F is any element of C, is the domain of C.

In particular, an element F of Φ is the distribution defined by all the restrictions of F. The distribution C' is called restriction of C (or $C' < C$) if C' is contained in C. Given a family $(F_i)_{i \in I}$ of compatible elements of Φ, there is a smallest distribution C such that $F_i < C$ for any i in I; C is called the distribution generated by $(F_i)_{i \in I}$.

A distribution C such that $\Omega(C) = R^n$ determines a linear functional also denoted by C on the set $\mathfrak{D}$ of all functions ϕ vanishing outside a bounded set. In this case, the notion is equivalent with that of a distribution defined by Schwartz (e) as an element of the dual of the vectorial topological space $\mathfrak{D}$. The definition given above is a particular case of the definition of distructures (c). Some methods which will be used are of local nature; it would be difficult to justify them with the ordinary (global) definition of distributions, but they follow easily from the theory of Ehresmann local structures.

Examples: A locally integrable function f defines a distribution still denoted by f (f). The distribution generated in R^1 by $d_x Z$, where:

$$\begin{cases} Z(x) = m & \text{if } x < m \\ Z(x) = x + m & \text{if } m \leqslant x, \end{cases}$$

is the Heaviside function Y_m. Its first functional derivative is the Dirac measure δ_m such that: $\delta_m(\phi) = \phi(m)$ for every ϕ.

Operations: We will use the following operations, where C denotes a distribution on R^n.

1) Dérivation:

$$\partial_{i_1 \ldots i_p} C(\phi) = (-1)^p C(\partial_{i_1 \ldots i_p} \phi)$$

2) Tensor product: If ϕ' is defined on R^{n+m} and vanishes outside a bounded set, let ϕ'_y be the function:

$$x \to \phi'(x, y)$$

where x is a point of R^n, y a point of R^m, so that (x, y) is a point of $R^{n+m} = R^n \times R^m$. The mapping ϕ'_C: $y \to C(\phi'_y)$ is an infinitely differentiable function on R^n which vanishes outside a bounded set. Let C' be a distribution on R^m; then, there is a distribution $C \otimes C'$ on R^{n+m} such that:

$$C \otimes C' \ (\phi') = C' \ (\phi'_C).$$

$C \otimes C'$ is called the tensor product of C and C' and we have: $C \otimes C' = C' \otimes C$.

3) Product by an infinitely differentiable function α:

$$\alpha C(\phi) = C(\alpha\phi).$$

4) Division by a polynomial: For every polynomial P, there exist distributions C/P such that:

$$(P(C/P))\ (\phi) = C(\phi).$$

All these distributions have the same restriction to the open set of R^n formed by the points which are not roots of P.

Temperate distributions: Since the definition of distributions is of a local nature in R^n, it can be extended to a manifold (that is a space which is locally homeomorphic to an open set of R^n, see Ehresmann [d]). In particular, some distributions on R^n are the restriction to R^n of a distribution the domain of which is the sphere S^n (obtained by adjoining to R^n a point ∞ at infinity). Such a distribution T is called a temperate distribution; T is a linear functional on the set $\mathcal{S}$ of all infinitely differentiable functions ψ which rapidly decrease to 0 near the point ∞, as well as every of their p-th partial derivatives.

The Fourier transform:

$$(\mathcal{F}\psi)(\xi) = -\frac{1}{2i\pi}\int_{-\infty}^{+\infty}\psi(x)\, e^{-2i\pi x.\xi}\, dx\ ,$$

where $i = \sqrt{-1}$, $x.\xi = x_1\xi_1 + \dots + x_n\xi_n$, ξ being a point of R^n (or rather of its dual space), is defined for every ψ. The Fourier transform of a temperate distribution T is the functional defined by:

$$(\mathcal{F}T)(\psi) = T(\mathcal{F}\psi),$$

for every ψ in $\mathcal{S}$; $\mathcal{F}T$ will also be denoted by T.

The distribution defined by a function f can be temperate as a distribution, even if the ordinary Fourier transform of f is not defined.

For every temperate distribution T, there exists an unique distribution $\hat{T}$ such that: $\mathcal{F}\hat{T} = T$. Writing $\hat{T} = \bar{\mathcal{F}}T$, we have: $\mathcal{F}(\bar{\mathcal{F}}T) = T$.

We will use the following formulas, in which δ_m is the Dirac measure on R^1, T a temperate distribution on R^n and ξ a point of R^1 (respectively of R^n):

$$\mathcal{F}\delta_m = e^{-2i\pi m\xi} \qquad (11)$$

$$\mathcal{F}(\partial_{j_1\dots j_p}T) = (2i\pi)^p\ \partial_{j_1\dots j_p}\hat{T} \qquad (12)$$

METHOD IN THE p-th LAYER

In this section, we will consider only the p-th layer and may omit the exponents p.

Problem (I): To find a solution of the system (1) satisfying the following boundary conditions:

(I_1) For t = 0, we have:

$$u_j \equiv 0,\ \frac{\partial u_j}{\partial t} \equiv 0, \text{ where } j = 1, 2, 3.$$

(I_2) For $t > 0$, the functions u_j are known functions $\underline{u}_j$ (resp. $\underline{\underline{u}}_j$) of t, x_1 and x_2, on the surface $x_3 = h_{p-1}$ (resp. $x_3 = h_p$), where j = 1, 2, 3.

As the system (1) is hyperbolic in the sense of Petrowsky [g], it is known that the problem (I) has one and only one solution corresponding to the given functions $\underline{u}_j$ and $\underline{\underline{u}}_j$.

The problem (I) being a mixed Cauchy problem, we apply to it the method indicated in [b]. This method consists in solving the problem (II):

Problem (II): To find a solution of the system (1) satisfying the conditions (I_1), (I_2), and the condition: (I_3) For $t > 0$, the functions $\partial_3 u_j$ are known functions $\underline{\partial_3 u_j}$ (resp. $\underline{\underline{\partial_3 u_j}}$) of t, x_1, x_2 on the surface $x_3 = h_{p-1}$ (resp. $x_3 = h_p$), where j = 1, 2, 3.

The condition (I_3) being in excess, the problem (II) has generally no solution; but if it admits a solution, then this solution is solution of the problem (I). Thus, by solving the problem (II), we will at once solve the problem (I) and find relations between $\underline{u}_j$, $\underline{\underline{u}}_j$, $\underline{\partial_3 u_j}$ and $\underline{\underline{\partial_3 u_j}}$.

Solution of the problem (II):

If u_j is solution of the problem (II), the functions v_j, j = 1, 2, 3, such that:

$$v_j \equiv 0 \quad \text{if } x_3 < h_{p-1} \text{ or } h_p < x_3$$
$$v_j \equiv u_j \quad \text{if } h_{p-1} \leqslant x_3 \leqslant h_p$$

are two times differentiable functions with respect to t, x_1, x_2, and have first order discontinuities with respect to x_3 on the surfaces $x_3 = h_{p-1}$ and $x_3 = h_p$. Considered as distributions, they have functional 3-partial derivatives such that:

$$\partial_3 v_j(\varphi) = -\int_{h_{p-1}}^{h_p} u_j(x)\ \partial_3\varphi(x)\, dx$$
$$= -(u_j(h_p)\varphi(h_p)) + (u_j(h_{p-1})\varphi(h_{p-1})) + \partial_3 u_j(\varphi)$$

i.e. $$\partial_3 v_j = \partial_3 u_j + \underline{u}_j \otimes \delta_{h_{p-1}} - \underline{\underline{u}}_j \otimes \delta_{h_p} \qquad (13)$$

and also:

$$\partial_{3k}v_j = \partial_{3k}u_j + \partial_k a_j\ ,\quad k = 1, 2$$
$$\partial_{33}v_j = \partial_{33}u_j + \partial_3 a_j + b_j \qquad (14)$$

where:

$$a_j = \underline{u}_j \otimes \delta_{h_{p-1}} - \underline{\underline{u}}_j \otimes \delta_{h_p}$$
$$b_j = \underline{\partial_3 u_j} \otimes \delta_{h_{p-1}} - \underline{\underline{\partial_3 u_j}} \otimes \delta_{h_p} \qquad (C_1)$$

From the equations (13), (14) and (1), we get the system:

$$\rho^2 \frac{\partial^2}{\partial t^2} v_j - \mu \Delta v_j - (\lambda+\mu) \partial_j \theta_V - h'_j \qquad (15)$$

where:

$$h'_j = -\mu(\partial_3 a_j + b_j) - (\lambda+\mu)\partial_j a_3 , \quad j = 1, 2$$
$$h'_3 = -(\lambda+2\mu)(\partial_3 a_3 + b_3) - (\lambda+\mu)(\partial_1 a_1 + \partial_2 a_2)$$

As the system (1) is hyperbolic, it can be proved (e) that the system (15) will only have for solutions the distributions v_j corresponding to the functions u_j solutions of (1); moreover if the functions $\underline{u}_j$ and $\underline{u}_j$ define temperate distributions, these solutions are also temperate distributions. Our method does not depend upon these results and could prove them anew (as done in (c)). We use them here for the sake of simplification.

Applying the Fourier transformation to the two members of (15), we obtain a system of linear equations, where $\tau, \xi_1, \xi_2, \xi_3$ denote the variables after the Fourier transformation:

$$\mu M^2 \hat{v}_j - (\lambda+\mu)\xi_j(\xi_1 \hat{v}_1 + \xi_2 \hat{v}_2 + \xi_3 \hat{v}_3) = -\frac{\hat{h}'_j}{4\pi^2} \qquad (16)$$
$$\hat{h}_j = -\mu(2i\pi\xi_3 \hat{a}_j + \hat{b}_j) - 2(\lambda+\mu) i\pi \xi_j \hat{a}_3 , \quad j = 1, 2$$
$$\hat{h}_3 = -(\lambda+2\mu)(2i\pi\xi_3 \hat{a}_3 + \hat{b}_3) - 2(\lambda+\mu) i\pi (\xi_1 \hat{a}_1 + \xi_2 \hat{a}_2)$$

where:

$$\left.\begin{aligned} \mu M^2 &= \rho\tau^2 - \mu(\xi_1^2 + \xi_2^2 + \xi_3^2) \\ (\lambda+2\mu) N^2 &= \rho\tau^2 - (\lambda+2\mu)(\xi_1^2 + \xi_2^2 + \xi_3^2) \end{aligned}\right| \qquad (C_2)$$

The ordinary Cramer method for linear equations reduces the system (16) to the system (17)* (the equations with * will be found at the end of this paper). If the problem (II) has a solution u_j, the corresponding distribution v_j admits as a Fourier transform one of the distributions obtained by dividing the second members of (17)* by the polynomial: $-2i\pi\mu(\lambda + 2\mu)M^2N^2$. Among the distributions obtained by this division, there is only one temperate distribution the inverse Fourier transform of which has a restriction equal to 0 on the set $x_3 < h_{p-1}$, $x_3 > h_p$ (see (c)). The transition from (1) to (15) may introduce solutions of (15) which are not solutions of the problem (II); in fact, the distributions introduced by the boundary conditions are "supported" by the boundary of the volume defined by:

$$t \geqslant 0, \quad h_{p-1} \leqslant x_3 \leqslant h_p$$

in the 4-dimensional space (t, x_1, x_2, x_3); this boundary has singular lines for $t = 0$, so that other boundary conditions could give rise to the same distributions. By expressing that the conditions in problem (I) are satisfied, we will determine necessary and sufficient relations between the given functions in problem (II) for the existence of a solution.

In order to write that the condition (I_2) is fulfilled, we are going to compute the inverse Fourier transforms $\check{\mathcal{F}}_{\xi_3} \hat{v}_j$ of the distributions $\hat{v}_j$ considered as distributions with respect to ξ_3 alone. The compete justification of the following computation, which requires a precise local study of the involved distributions, can be found in (c); this is one of the most difficult parts of our problem.

<u>Preliminary inverse Fourier transforms</u>:

Let us first compute:

$$I = \check{\mathcal{F}}_{\xi_3} (\mathrm{Pf}(1/\mu M^2)),$$

where $\mathrm{Pf}(1/\mu M^2)$ is the particular distribution obtained by division of 1 by μM^2 the inverse Fourier transform of which is:

$$\frac{-4\pi^2 Z_2}{\rho^{1/2}\mu^{3/2}}$$

with respect to $(\tau, \xi_1, \xi_2, \xi_3)$; Z_2 is the distribution of M. Riesz in R^4 defined (h) by:

$$Z_2 = \lim_{l \to 2} \frac{s^{l-4}}{2^{l-1} \pi \Gamma(\frac{l}{2}) \Gamma(\frac{l-2}{2})}$$

$$\left| \begin{array}{ll} s^2 = t^2/\rho - r^2/\mu & \text{if } t^2/\rho \geqslant r^2/\mu \text{ and } t \geqslant 0 \\ s^2 = 0 & \text{otherwise} \end{array} \right.$$
$$r = \sqrt{x_1^2 + x_2^2 + x_3^2} \qquad (C_3)$$

Since Z_2 is a temperate distribution, we have also:

$$I = \frac{-4\pi^2}{\rho^{1/2}\mu^{3/2}} \mathcal{F}_{x_1, x_2, t} Z_2$$

Let ϵ be a positive number; it may be proved that:

$$I = \lim_{\varepsilon \to 0} \left(-\frac{4\pi^2}{\rho^{1/2}\mu^{3/2}} \int e^{-2i\pi(x_1\xi_1 + x_2\xi_2)} I' dx \right) \quad (18)$$

where:

$$I' = \int_{\sqrt{\rho/\mu}\, r}^{\infty} Z_2 \, e^{-2i\pi t(\tau - i\varepsilon)} \, dt$$
$$= \lim_{l \to 2} \frac{1}{2^{l-1} \pi \Gamma(\frac{l}{2}) \Gamma(\frac{l-2}{2})} (\mathcal{L}\, s^{l-4})(\tau_0)$$
$$\tau_0 = 2\pi(\varepsilon - i\tau) .$$

The symbol $\mathcal{L}$ denotes the ordinary Laplace transformation. We have:

$$(\mathcal{L}\, s^{l-4})(\tau_0) = \frac{\Gamma(\frac{l-2}{2})}{\rho^{\frac{l-5}{4}} \pi^{\frac{1}{2}}} \left(\frac{2r}{\sqrt{\mu}\,\tau_0} \right)^{\frac{l-3}{2}} K_{\frac{l-3}{2}}\left(\sqrt{\frac{\rho}{\mu}}\, \tau r \right) \quad (19)$$

K_ν being the modified Hankel function (i); since:

$$K_{-1/2}(t) = \sqrt{\frac{\pi}{2t}}\, e^{-t}$$

we obtain:

$$I' = \frac{\sqrt{\rho\mu}\; e^{-\sqrt{\frac{\rho}{\mu}}\, \tau_0 r}}{4\pi r} \qquad (20)$$

We apply Bochner's formula (j):

$$\hat{f}(|\xi|) = \frac{2\pi}{|\xi|^{\frac{n-2}{2}}} \int_0^\infty f(y)\, y^{n/2} I_{\frac{n-2}{2}}(2\pi|\xi| y)\, dy$$

in which f denotes a function of $|x| = y = \sqrt{x_1^2 + \ldots + x_n^2}$ in R^n, $|\xi| = \sqrt{\xi_1^2 + \ldots + \xi_n^2}$, and I_0 is modified Bessel function (J), in the case n = 2. From (18) and (20) it follows then:

$$I = \lim_{\epsilon \to 0} \left[\frac{-2\pi^2}{\mu} \int_0^{+\infty} y \, e^{-\tau_0 \sqrt{\frac{\rho}{\mu}(y^2 + x_3^2)}} \, I_0(2\pi y d) \, dy \right.$$

$$d^2 = \xi_1^2 + \xi_2^2 \qquad (C_4)$$

This integral is the limit for $\epsilon \to 0$ of the Fourier transform in u = 0, of the function:

$$\left| \begin{array}{ll} u \longrightarrow e^{-(\rho/\mu)^{\frac{1}{2}} \tau_0 u} & I_0(2\pi d(u^2 - x_3^2)^{\frac{1}{2}}) \\ & \text{if } |x_3| < u \\ u \longrightarrow 0 & \text{otherwise} \end{array} \right.$$

where $u = \sqrt{y^2 + x_3^2}$.

From this, we get:

$$AI = -(\pi/\mu) e^{-2\pi |x_3| A} \qquad (21)$$

where:

$$\left. \begin{array}{ll} A = \sqrt{d^2 - \frac{\rho\tau^2}{\mu}} & \text{if } \mu d^2 \geq \rho\tau^2 \\ A = i\sqrt{\frac{\rho\tau^2}{\mu} - d^2} & \text{if } \mu d^2 < \rho\tau^2 \end{array} \right| (C_5)$$

In the same way, we have:

$$B \, \check{\mathcal{F}}_{\xi_3} \left(Pf \frac{1}{(\lambda + 2\mu) N^2}\right)(x_3) = \frac{-\pi}{\lambda + 2\mu} e^{-2\pi |x_3| B} \qquad (22)$$

where:

$$\left. \begin{array}{ll} B = \sqrt{d^2 - \frac{\rho\tau^2}{\lambda + 2\mu}} & \text{if } (\lambda + 2\mu) d^2 \geq \rho\tau^2 \\ B = i\sqrt{\frac{\rho\tau^2}{\lambda + 2\mu} - d^2} & \text{if } (\lambda + 2\mu) d^2 < \rho\tau^2 \end{array} \right| (C_6)$$

Furthermore:

$$T = Pf \frac{1}{\mu(\lambda + 2\mu) M^2 N^2} = Pf \left[\frac{1}{(\lambda + \mu)\tau^2} \left(\frac{1}{N^2} - \frac{1}{M^2} \right) \right]$$

and:

$$(A^2 - B^2) AB \, \check{\mathcal{F}}_{\xi_3} T = \pi \left[A e^{-2\pi |x_3| B} - B e^{-2\pi |x_3| A} \right] \qquad (23)$$

Inverse Fourier transform of $\hat{v}_j$:

$\hat{v}_j$ is a sum of terms of the form:

$$S = Pf\left(\frac{\alpha \mathcal{F}(T \otimes \delta_m)}{M^2 N^2} \right) = Pf\left(\frac{\alpha \mathcal{F} T \otimes e^{-2i\pi m \xi_3}}{M^2 N^2} \right)$$

where T is a known distribution, α a polynomial; Pf denotes the quotient distribution such that the restriction of its inverse Fourier transform to the set: $x_3 < h_{p-1}$ or $h_p < x_3$ is 0.

We will not prove here the following result, the demonstration of which depends essentially upon the local definition of a distribution and may be found in (c): $\check{\mathcal{F}}_{\xi_3} S$ is in fact a function of x_3 taking its values in the space of the distributions with respect to (τ, ξ_1, ξ_2) and such that:

$$(\check{\mathcal{F}}_{\xi_3} S)(x_3) = (\check{\mathcal{F}}_{\xi_3} (\frac{\alpha}{M^2 N^2}))(x_3 - m) \, \hat{T}.$$

The distributions $\check{\mathcal{F}}_{\xi_3}(\frac{\alpha}{N^2 M^2})$ are obtained from the formulas (12), (21), (22) and (23). Thus, the Fourier transform of $\hat{v}_j$ with respect to ξ_3 is a function of x_3, sum of functions $\check{\mathcal{F}}_{\xi_3} S$.

Let us write the equations:

$$\xi_1 (\check{\mathcal{F}}_{\xi_3} \hat{v}_1)(h_{p-1}) + \xi_2 (\check{\mathcal{F}}_{\xi_3} \hat{v}_2)(h_{p-1}) = \xi_1 \hat{\underline{u}}_1 + \xi_2 \hat{\underline{u}}_2$$

$$(\check{\mathcal{F}}_{\xi_3} \hat{v}_3)(h_{p-1}) = \hat{\underline{u}}_3$$

which express the condition (I_2). After multiplication of the two members of these equations by: $\mu(\lambda + 2\mu) AB(A^2 - B^2)$ we obtain the system (24)*, in which we have put:

$$\left. \begin{array}{l} \underline{W} = \xi_1 \hat{\underline{u}}_1 + \xi_2 \hat{\underline{u}}_2, \quad \underline{w} = \hat{\underline{u}}_3 \\ \underline{\underline{W}} = \xi_1 \hat{\underline{\underline{u}}}_1 + \xi_2 \hat{\underline{\underline{u}}}_2, \quad \underline{\underline{w}} = \hat{\underline{\underline{u}}}_3 \\ 2i\pi \underline{\Phi} = \xi_1 \mathcal{F} \underline{\partial_3 u_1} + \xi_2 \mathcal{F} \underline{\partial_3 u_2}, \quad 2i\pi \underline{\varphi} = \mathcal{F} \underline{\partial_3 u_3} \\ 2i\pi \underline{\underline{\Phi}} = \xi_1 \mathcal{F} \underline{\underline{\partial_3 u_1}} + \xi_2 \mathcal{F} \underline{\underline{\partial_3 u_2}}, \quad 2i\pi \underline{\underline{\varphi}} = \mathcal{F} \underline{\underline{\partial_3 u_3}} \end{array} \right| (C_7)$$

$$a = e^{-2\pi(h_p - h_{p-1})A}; \quad b = e^{-2\pi(h_p - h_{p-1})B} \qquad (C_8)$$

The system (24)* indicates the relations between the given functions $\underline{W}, \underline{w}, \underline{\underline{W}}, \underline{\underline{w}}, \underline{\Phi}, \underline{\underline{\Phi}}, \underline{\varphi}, \underline{\underline{\varphi}}$, which are necessary and sufficient for the existence of a solution of problem (II). By the ordinary Cramer method for linear equations, we obtain from (24)* the systems (25)*, (26)* and (27)*. In these systems, we have put:

$$\begin{array}{l} c^K = 1 - e^{-2\pi(h_p - h_{p-1})K} \\ c_K = 1 + e^{-2\pi(h_p - h_{p-1})K} \end{array} \qquad (C_9)$$

where K = A, B, 2A, 2B, with the following convention:

$$c^L c_{L'} = c^L_{L'}, \; c_L c_{L'} = c_{LL'}, \; c^L c^{L'} = c^{LL'}$$

By replacing in (17)* $\underline{\Phi}, \underline{\underline{\Phi}}, \underline{\varphi}$ and $\underline{\underline{\varphi}}$ by their values given by (26)*, we find the distributions $\hat{v}_j$, the Fourier transforms of which are the distributions solutions of the problem (I). If in (25)* we apply the transformation:

$\underline{W} \leftrightarrow \underline{\underline{W}}$, $\underline{w} \leftrightarrow -\underline{\underline{w}}$, $\underline{\Phi} \leftrightarrow -\underline{\underline{\Phi}}$, $\underline{\varphi} \leftrightarrow \underline{\underline{\varphi}}$ we get $(\underline{\underline{W}}, \underline{\underline{w}})$ as functions of $(\underline{W}, \underline{w})$ and $(\underline{\Phi}, \underline{\varphi})$.

Last layer

The method is the same as in the p-th layer, but the boundary surface is $x_3 = h_{q-1}$; the conditions (I_2) and (I_3) are now: (I_2^q) For $t > 0$, the functions u_j^q are known functions $\underline{u}_j^q$ of (t, x_1, x_2) on the surface $x_3 = h_{q-1}$, where $j = 1, 2, 3$. (I_3^q) For $t > 0$, the functions $\partial_3 u_j^q$ are known functions $\underline{\partial_3 u_j^q}$ of (t, x_1, x_2) on surface $x_3 = h_{q-1}$, where $j = 1, 2, 3$.

The equations (17) are still verified if we replace a_j and b_j by:

$$a_j^q = \underline{u}_j^q \otimes \delta_{h_{q-1}}, \; b_j^q = \underline{\partial_3 u_j^q} \otimes \delta_{h_{q-1}} \quad (C_1^q).$$

$$\left.\begin{aligned}
-2i\pi\,\mu(\lambda+2\mu)M^2N^2\hat{v}_j &= \begin{bmatrix}\mu(\lambda+2\mu)\xi_3 N^2\hat{a}_j+(\lambda+\mu)(\lambda+2\mu)\xi_j\xi_3(\xi_1\hat{a}_1+\xi_2\hat{a}_2)+(\lambda+\mu)\xi_j(M^2+\xi_3^2)\hat{a}_3\\ +2i\pi\left[\mu(\lambda+2\mu)N^2\hat{b}_j+(\lambda+\mu)\xi_j(\mu(\xi_1\hat{b}_1+\xi_2\hat{b}_2)+(\lambda+2\mu)\xi_3\hat{b}_3)\right]\end{bmatrix} \quad j=1,2\\
-2i\pi\,\mu(\lambda+2\mu)M^2N^2\hat{v}_3 &= \begin{bmatrix}(\lambda+\mu)(\lambda+2\mu)(N^2+\xi_3^2)(\xi_1\hat{a}_1+\xi_2\hat{a}_2)+\mu\xi_3\left[(\lambda+2\mu)M^2-(\lambda+\mu)(\xi_1^2+\xi_2^2)\right]\hat{a}_3\\ +2i\pi\left[\mu(\lambda+\mu)\xi_3(\xi_1\hat{b}_1+\xi_2\hat{b}_2)+(\lambda+2\mu)\left[(\lambda+2\mu)N^2+(\lambda+\mu)\xi_3^2\right]\hat{b}_3\right.\end{bmatrix}
\end{aligned}\right| \quad (17)^*$$

$$\left.\begin{aligned}
&-i\mu(\lambda+2\mu)(A^2-B^2)B\underline{W}+\mu(\lambda+\mu)d^2A(A-B)\underline{w}+i(\lambda+\mu)(\lambda+2\mu)B(B^2a-d^2b)\underline{\underline{W}}+\mu(\lambda+\mu)d^2A(Ba-Ab)\underline{\underline{w}}=\\
&\qquad -\mu(\lambda+\mu)(AB-d^2)\underline{\Phi}-\mu(\lambda+\mu)(d^2b-ABa)\underline{\underline{\Phi}}+id^2(\lambda+\mu)(\lambda+2\mu)B(b-a)\underline{\underline{\varphi}}\\
&(\lambda+\mu)(\lambda+2\mu)B(A-B)\underline{W}+i\mu(\lambda+2\mu)(B^2-A^2)A\underline{w}-(\lambda+2\mu)(\lambda+\mu)B(Ab-Ba)\underline{\underline{W}}+i\mu(\lambda+\mu)A(A^2b-d^2a)\underline{\underline{w}}=\\
&\qquad -(\lambda+2\mu)(\lambda+\mu)(AB-d^2)\underline{\varphi}+i\mu(\lambda+\mu)A(b-a)\underline{\underline{\Phi}}-(\lambda+2\mu)(\lambda+\mu)(d^2a-ABb)\underline{\underline{\varphi}}
\end{aligned}\right| \quad (24)^*$$

$$2\rho\tau^2abB\underline{\underline{W}}=-i\mu(ABc^{2A}b-d^2c^{2B}a)\underline{\Phi}+(\lambda+2\mu)Bd^2(c_{2B}a-c_{2A}b)\underline{\varphi}+(\lambda+2\mu)B(d^2c_{2B}a-B^2c_{2A}b)\underline{W}+i\mu Ad^2(Bc^{2A}b-Ac^{2B}a)\underline{w}$$

$$2\rho\tau^2abA\underline{\underline{w}}=-\mu A(c_{2A}b-c_{2B}a)\underline{\Phi}-i(\lambda+2\mu)(ABc^{2B}a-d^2c^{2A}b)\underline{\varphi}-i(\lambda+2\mu)B(Ac^{2B}a-Bc^{2A}b)\underline{W}+\mu A(d^2c_{2A}b-A^2c_{2B}a)\underline{w} \quad (25)^*$$

$$\left.\begin{aligned}
\mu(ABc_B^A-d^2c_A^B)(\underline{\Phi}+\underline{\underline{\Phi}}) &= -i\rho\tau^2Bc_{AB}(\underline{W}-\underline{\underline{W}})-\mu d^2A(Ac_A^B-Bc_B^A)(\underline{w}+\underline{\underline{w}})\\
\mu(ABc_A^B-d^2c_B^A)(\underline{\Phi}-\underline{\underline{\Phi}}) &= -i\rho\tau^2Bc^{AB}(\underline{W}+\underline{\underline{W}})-\mu d^2A(Ac_B^A-Bc_A^B)(\underline{w}-\underline{\underline{w}})\\
(\lambda+2\mu)(d^2c_B^A-ABc_A^B)(\underline{\varphi}+\underline{\underline{\varphi}}) &= -(\lambda+2\mu)B(Bc_B^A-Ac_A^B)(\underline{W}+\underline{\underline{W}})+i\rho\tau^2Ac_{AB}(\underline{w}-\underline{\underline{w}})\\
(\lambda+2\mu)(ABc_B^A-d^2c_A^B)(\underline{\varphi}-\underline{\underline{\varphi}}) &= (\lambda+2\mu)B(Bc_A^B-Ac_B^A)(\underline{W}-\underline{\underline{W}})-i\rho\tau^2Ac^{AB}(\underline{w}+\underline{\underline{w}})
\end{aligned}\right| \quad (26)^*$$

$$\left.\begin{aligned}
2\rho\tau^2a^1b^1B^1\underline{W}^2 &= \begin{bmatrix}-i(A^1B^1c^{2A^1}b^1-d^2c^{2B^1}a^1)\Psi+d^2B^1(c_{2B^1}a^1-c_{2A^1}b^1)\psi\\ +B^1\left[2\mu^1d^2(c_{2B^1}a^1-c_{2A^1}b^1)+\rho^1\tau^2b^1c_{2A^1}\right]\underline{W}^1+id^2\left[2\mu^1(A^1B^1c^{2A^1}b^1-d^2c^{2B^1}a^1)+\rho\tau^2c^{2B^1}a^1\right]\underline{w}^1\end{bmatrix}\\
2\rho\tau^2a^1b^1A^1\underline{w}^2 &= \begin{bmatrix}A^1(c_{2B^1}a^1-c_{2A^1}b^1)\Psi-i(A^1B^1c^{2B^1}a^1-d^2c^{2A^1}b^1)\psi\\ -i\left[2\mu^1(A^1B^1c^{2B^1}a^1-d^2c^{2A^1}b^1)+\rho\tau^2c^{2A^1}b^1\right]\underline{W}^1+A^1\left[2\mu^1d^2(c_{2A^1}b^1-c_{2B^1}a^1)+\rho\tau^2a^1c_{2B^1}\right]\underline{w}^1\end{bmatrix}
\end{aligned}\right| \quad (31)^*$$

$$\left.\begin{aligned}
&2\mu^2\rho\tau^2a^1b^1A^1\underline{\Phi}^2=\\
&\begin{bmatrix}A^1\left[(2\mu^1-\mu^2)d^2(c_{2B^1}a^1-c_{2A^1}b^1)+\rho\tau^2c_{2A^1}b^1\right]\Psi-id^2\left[(2\mu^1-\mu^2)(A^1B^1c^{2B^1}a^1-d^2c^{2A^1}b^1)+\rho\tau^2b^1c^{2A^1}\right]\psi\\ -i\left[2\mu^1(2\mu^1-\mu^2)d^2(A^1B^1c^{2B^1}a^1-(A^1)^2c^{2A^1}b^1)+\rho\tau^2c^{2A^1}(A^1)^2\right]\underline{W}^1+d^2A^1\left[\mu^1(2\mu^1-\mu^2)((A^1)^2+d^2)(c_{2A^1}b^1-c_{2B^1}a^1)-\mu^2\rho\tau^2c_{2A^1}b^1\right]\underline{w}^1\end{bmatrix}\\
&2(\lambda^2+2\mu^2)\rho\tau^2a^1b^1B^1\underline{\varphi}^2=\\
&\begin{bmatrix}i\left[(2\mu^1+\lambda^1)(A^1B^1c^{2A^1}b^1-d^2c^{2B^1}a^1)+\rho\tau^2a^1c^{2B^1}\right]\Psi+B^1\left[(2\mu^1+\lambda^1)d^2(c_{2A^1}b^1-c_{2B^1}a^1)+\rho\tau^2c_{2B^1}a^1\right]\psi\\ -B^1\left[\mu^1(2\mu^1+\lambda^2)((A^1)^2+d^2)(c_{2B^1}a^1-c_{2A^1}b^1)+\lambda^2\rho\tau^2c_{2B^1}a^1\right]\underline{W}^1-i\left[2\mu^1(2\mu^1+\lambda^2)d^2(A^1B^1c^{2A^1}b^1-(A^1)^2c^{2B^1}a^1)+\rho\tau^2(\lambda d^2-\rho\tau^2)c^{2B^1}a^1\right]\underline{w}^1\end{bmatrix}
\end{aligned}\right| \quad (32)^*$$

$$2\rho\tau^2A^pa^pb^p\mu^{p+1}\underline{\Phi}^{p+1}= \quad (34)_1^*$$

$$\begin{bmatrix}+\mu^pA^p\left[(2\mu^p-\mu^{p+1})d^2(c_{2B^p}a^p-c_{2A^p}b^p)+\rho\tau^2c_{2A^p}b^p\right]\underline{\Phi}^p-i(\lambda^p+2\mu^p)d^2\left[(2\mu^p-\mu^{p+1})(A^pB^pc^{2B^p}a^p-d^2c^{2A^p}b^p)+\rho\tau^2c^{2A^p}b^p\right]\underline{\varphi}^p\\ -i(\lambda^p+2\mu^p)B^p\left[(2\mu^p-\mu^{p+1})d^2(A^pc^{2B^p}a^p-B^pc^{2A^p}b^p)+\rho\tau^2c^{2A^p}b^pB^p\right]\underline{W}^p+\mu^pd^2A^p\left[(2\mu^p-\mu^{p+1})(d^2c_{2A^p}b^p-(A^p)^2c_{2B^p}a^p)-\rho\tau^2c_{2A^p}b^p\right]\underline{w}^p\end{bmatrix}$$

$$2(\lambda^{p+1}+2\mu^{p+1})\rho\tau^2B^pa^pb^p\underline{\varphi}^{p+1}= \quad (34)_2^*$$

$$\begin{bmatrix}+i\mu^p\left[(2\mu^p+\lambda^{p+1})(A^pB^pc^{2A^p}b^p-d^2c^{2B^p}a^p)+\rho\tau^2c^{2B^p}a^p\right]\underline{\Phi}^p+(\lambda^p+2\mu^p)A^p\left[(2\mu^p+\lambda^{p+1})d^2(c_{2A^p}b^p-c_{2B^p}a^p)+\rho\tau^2c_{2B^p}a^p\right]\underline{\varphi}^p\\ -(\lambda^p+2\mu^p)B^p\left[(2\mu^p+\lambda^{p+1})(d^2c_{2B^p}a^p-(B^p)^2c_{2A^p}b^p)-\rho\tau^2c_{2B^p}a^p\right]\underline{W}^p-i\mu^pA^p\left[(2\mu^p+\lambda^{p+1})d^2(B^pc^{2A^p}b^p-A^pc^{2B^p}a^p)+\rho\tau^2c^{2B^p}a^pA^p\right]\underline{w}^p\end{bmatrix}$$

$$A^1 D_{A^1} = \left\{ 2(\mu^1-\mu^2) d^2 \left[A^1 a^1 (B^1 c^{2B^1 2B^2} - B^2 c_{2B^1 2B^2}) + b^1 (A^1 B^2 c_{2A^1 2B^2} - d^2 c^{2A^1 2B^2}) \right] + \rho\tau^2 a^1 A^1 (B^1 c^{2B^1 2B^2} + B^2 c_{2B^1 2B^2}) \right\} a^2 \rho\tau^2$$

$$A^1 E_{A^1} = 2(\mu^2-\mu^1) a^1 b^2 \rho^2 \tau^4 (A^1 B^2 c_{2A^1 2B^2} - d^2 c^{2A^1 2B^2})$$

$$B^1 F_{A^1} = \rho\tau^2 \left\{ 2(\mu^2-\mu^1) a^2 \left[d^2 a^1 (B^1 c^{2B^2}_{2B^1} - B^2 c^{2B^1}_{2B^2}) + b^1 B^1 (A^1 B^2 c^{2A^1}_{2B^1} - d^2 c^{2B^2}_{2A^1}) \right] - \rho\tau^2 (B^1 c^{2B^2}_{2B^1} + B^2 c^{2B^1}_{2B^2}) a^1 a^2 \right\}$$

$$A^1 G_{A^1} = d^2 \begin{bmatrix} 2(\mu^2-\mu^1) \left[A^2 B^2 c^{2A^1} b^2 - d^2 c^{2B^2} a^2)(A^1 B^1 c^{2B^1} a^1 - d^2 c^{2A^1} b^1) - d^2 A^1 B^2 (c_{2B^1} a^1 - c_{2A^2} b^1)(c_{2A^1} b^2 - c_{2B^1} a^2) \right] \\ + \rho\tau^2 \left[A^1 a^1 a^2 (B^1 c^{2B^1 2B^2} + B^2 c_{2B^1 2B^2}) - B^2 b^1 b^2 (A^2 c^{2A^1 2A^2} + A^1 c_{2A^1 2A^2}) \right] \end{bmatrix}$$

$$B^1 H_{A^1} = \begin{bmatrix} 2(\mu^2-\mu^1) d^2 \left[B^1 (A^2 B^2 c^{2A^1} b^2 - d^2 c^{2B^2} a^2)(c_{2A^1} b^1 - c_{2B^1} a^1) - B^2 (A^1 B^1 c^{2A^1} b^1 - d^2 c^{2B^1} a^1)(c_{2A^2} b^2 - c_{2B^2} a^2) \right] \\ + \rho\tau^2 \left[(A^1 B^1 c^{2A^1}_{2A^2} b^1 b^2 - d^2 c^{2B^1}_{2B^2} a^1 a^2) B^2 + (A^2 B^2 c^{2A^1}_{2A^1} b^1 b^2 - d^2 c^{2B^2}_{2B^1} a^1 a^2) B^1 \right. \end{bmatrix}$$

$$B^1 K_{A^1} = \rho^2 \tau^4 a^1 a^2 \left[\rho\tau^2 (B^2 c^{2B^1}_{2B^2} + B^1 c^{2B^2}_{2B^1}) - 2(\mu^1-\mu^2) d^2 (B^2 c^{2B^1}_{2B^2} - B^1 c^{2B^2}_{2B^1}) \right] \qquad (C_{11})^*$$

$$4\rho^2\tau^4 a^1 b^1 a^2 b^2 A^1 B^1 B^2 \underline{W}^3 = i A^1 B^1 H_{A^1} \Psi + d^2 A^1 B^1 G_{A^1} \psi + B^1 (2\mu^1 A^1 G_{A^1} + B^2 D_{B^2}) \underline{W}^1 + i d^2 A^1 (2\mu^1 B^1 H_{A^1} - B^2 F_{A^1}) \underline{w}^1$$

$$4\rho^2\tau^4 a^1 b^1 a^2 b^2 A^1 A^2 B^1 \underline{w}^3 = - A^1 B^1 G_{B^1} \Psi - i A^1 B^1 H_{A^2} \psi - i d^2 B^1 (2\mu^1 A^1 H_{B^1} - A^2 F_{B^2}) \underline{W}^1 + A^1 (2\mu^1 B^1 G_{B^1} + A^2 D_{A^2}) \underline{w}^1 \qquad (35)^*$$

$$\mu^3 a^2 b^2 A^1 A^2 B^1 \underline{\underline{\Phi}}^3 =$$

$$= a^1 b^1 \begin{bmatrix} A^1 B^1 [(\mu^3 - 2\mu^2) G_{B^1} + D_{B^1}] \Psi + i d^2 A^1 B^1 [(\mu^3 - 2\mu^2) H_{B^1} + F_{B^1}] \psi \\ + i B^1 [(2\mu^2-\mu^3)(F_{B^2} A^2 - 2\mu^1 d^2 A^1 H_{B^1}) + (2\mu^1 F_{B^1} + K_{B^1}) A^2] \underline{W}^1 + d^2 A^1 [(2\mu^2-\mu^3)(2\mu^1 B^1 G_{B^1} + D_{A^1} B^2) - B^1 (2\mu^1 D_{B^1} + E_{B^1})] \underline{w}^1 \end{bmatrix}$$

$$(\lambda^3 + 2\mu^3) a^2 b^2 A^1 B^1 B^2 \underline{\underline{\varphi}}^3 =$$

$$= a^1 b^1 \begin{bmatrix} i A^1 B^1 [(\lambda^3 + 2\mu^2) H_{A^1} - F_{A^1}] \Psi + A^1 B^1 [-(\lambda^3 + 2\mu^2) G_{A^1} + D_{A^1}] \psi \\ - B^1 [(2\mu^2+\lambda^3)(2\mu^1 A^1 G_{A^1} + D_{B^2} A^2) - A^1 (2\mu^1 D_{A^1} + E_{A^1})] \underline{W}^1 - B^1 [(2\mu^2+\lambda^3)(F_{A^2} B^2 - 2\mu^1 d^2 B^1 H_{A^1}) + (2\mu^1 F_{A^1} + K_{A^1}) B^2] \underline{w}^1 \end{bmatrix} \qquad (36)^*$$

$$\mu^3 (A^4 B^4 - d^2) \underline{\underline{\Phi}}^3 = -i \rho^4 \tau^2 B^4 \underline{W}^4 + d^2 [(2\mu^4 - \mu^3)(A^4 B^4 - d^2) + \rho^4 \tau^2] \underline{w}^4$$

$$(\lambda^3 + 2\mu^3)(A^4 B^4 - d^2) \underline{\underline{\varphi}}^3 = -[(\lambda^3 + 2\mu^4)(A^4 B^4 - d^2) + \rho^4 \tau^2] \underline{W}^4 - i \rho^4 \tau^2 A^4 \underline{w}^4 \qquad (37)^*$$

$$J_A = 2(\mu^4 - \mu^3)(A^4 B^4 - d^2)(A^3 B^3 c^{2A^3} b^3 - d^2 c^{2B^3} a^3) + \rho\tau^2 [B^3 (A^3 c^{2A^3} + A^4 c_{2A^3}) b^3 - A^4 (B^3 c_{2B^3} + B^4 c^{2B^3}) a^3]$$

$$L_A = 2(\mu^3 - \mu^4)(A^4 B^4 - d^2) d^2 B^3 (c_{2B^3} a^3 - c_{2A^3} b^3) + \rho\tau^4 [B^3 B^4 (A^3 c^{2A^3} + A^4 c_{2A^3}) b^3 - d^2 (B^3 c_{2B^3} + B^4 c^{2B^3}) a^3]$$

$$Q_A = \rho\tau^2 a^3 [2(\mu^3 - \mu^4)(A^4 B^4 - d^2) c_{2B^3} B^3 - \rho\tau^2 (B^4 c^{2B^4} + B^3 c_{2B^3})]$$

$$T_A = \rho\tau^2 a^3 [2(\mu^3 - \mu^4)(A^4 B^4 - d^2) c^{2B^3} d^2 - \rho\tau^4 A^4 (B^4 c^{2B^4} + B^3 c_{2B^3})] \qquad (C_{12})^*$$

$$2\mu^3 \rho\tau^2 a^3 b^3 A^3 (A^4 B^4 - d^2) \underline{\Phi}^3 = i (T_B - J_A \mu^3 d^2) \underline{W}^4 + d^2 (\mu^3 L_B - Q_B) \underline{w}^4$$

$$2(\lambda^3 + 2\mu^3) \rho\tau^2 a^3 b^3 B^3 (A^4 B^4 - d^2) \underline{\varphi}^3 = -[(\lambda^3 + 2\mu^3) L_A - Q_A] \underline{W}^4 + i [T_A - (\lambda^3 + 2\mu^3) d^2 J_A] \underline{w}^4 \qquad (39)^*$$

$$a^3 b^3 A^3 (A^4 B^4 - d^2) [(2\mu^1 D_{A^1} + E_{A^1}) \underline{W}^1 + i (2\mu^1 d^2 F_{A^1} + K_{A^1}) \underline{w}^1] - 2 a^1 b^1 a^2 b^2 B^2 [(L_{A^1} + Q_{A^1}) \underline{W}^4 - i (J_{A^1} + T_{A^1}) \underline{w}^4] =$$
$$= a^3 b^3 (A^4 B^4 - d^2) B^3 [i F_{A^1} \Psi - D_{A^1} \psi]$$

$$a^3 b^3 A^3 (A^4 B^4 - d^2) [-i (2\mu^1 F_{B^1} d^2 + K_{B^1}) \underline{W}^1 + d^2 (2\mu^1 D_{B^1} + E_{B^1}) \underline{w}^1] - 2 a^1 b^1 a^2 b^2 A^2 [i (J_{B^1} + T_{B^1}) \underline{W}^4 - (L_{B^1} + Q_{B^1}) \underline{w}^4] =$$
$$= a^3 b^3 (A^4 B^4 - d^2) A^3 [D_{B^1} \Psi + i d^2 F_{B^1} \psi]$$

$$a^3 b^3 A^3 (A^4 B^4 - d^2) [-i (Z_{B^1} + 2\mu^1 d^2 F_{B^1} + K_{B^1}) \underline{W}^1] + d^2 (X_{B^1} + 2\mu^1 D_{B^1} + E_{B^1}) \underline{w}^1 + 2 a^1 b^1 a^2 b^2 A^2 [i T_{B^1} \underline{W}^4 - d^2 Q_{B^1} \underline{w}^4]$$
$$= a^3 b^3 (A^4 B^4 - d^2) A^3 \{ [2(\mu^3 - \mu^2) G_{B^1} + D_{B^1}] \Psi + i d^2 [2(\mu^3 - \mu^2) H_{B^1} + F_{B^1}] \psi \}$$

$$a^3 b^3 B^3 (A^4 B^4 - d^2) [(X_{A^1} + 2\mu^1 D_{A^1} + E_{A^1}) \underline{W}^1 + i (Z_{A^1} + 2\mu^1 d^2 F_{A^1} + K_{A^1}) \underline{w}^1] - 2 a^1 b^1 a^2 b^2 B^2 [Q_{A^1} \underline{W}^4 + i T_{A^1} \underline{w}^4] =$$
$$= a^3 b^3 (A^4 B^4 - d^2) B^3 \{ i [2(\mu^3 - \mu^2) H_{A^1} + F_{A^1}] \Psi - [2(\mu^3 - \mu^2) G_{A^2} + D_{A^1}] \psi \} \qquad (40)^*$$

$$2\pi\mu [(\lambda+2\mu)^2 (A+B)^2 (A-B) - 4(\lambda+\mu)^2 d^2 A] \hat{u}_j = (\lambda+\mu) [(\lambda+2\mu) A (A+B) \hat{\pi}_j + i \xi_j [(\lambda+2\mu) B - \lambda A] \hat{\pi}_3 \quad , \; j = 1,2$$

$$2\pi\mu [(\lambda+2\mu)^2 (A+B)^2 (A-B) - 4(\lambda+\mu)^2 d^2 A] \hat{u}_3 = (\lambda+\mu) [i [\lambda A - (\lambda+2\mu) B] (\xi_1 \hat{\pi}_1 + \xi_2 \hat{\pi}_2) + (\lambda+2\mu) B (A+B) \hat{\pi}_3 \qquad (42)^*$$

The condition (I_2^q) is expressed by the following system:

$$\begin{aligned}\underline{\Phi}^q\left[\mu^q A^q+(\lambda^q+2\mu^q)B^q\right] &= -i(\lambda^q+2\mu^q)(A^q+B^q)B^q\underline{W}^q-(\lambda^q+\mu^q)d^2\Lambda^q\underline{w}^q \\ \underline{\varphi}^q\left[\mu^q A^q+(\lambda^q+2\mu^q)B^q\right] &= -(\lambda^q+\mu^q)B^q\underline{W}^q+i\mu^q(A^q+B^q)A^q\underline{w}^q\end{aligned} \qquad (27)$$

where:

$$\begin{aligned}&\underline{W}^q = \xi_1\hat{\underline{u}}_1^q + \xi_2\hat{\underline{u}}_2^q, \quad \underline{w}^q=\hat{\underline{u}}_3^q \\ &2i\pi\,\underline{\Phi}^q=\xi_1\mathcal{F}(\underline{\partial_3 u_1^q})+\xi_2\mathcal{F}(\underline{\partial_3 u_2^q}), \quad 2i\pi\underline{\varphi}^q=\mathcal{F}(\underline{\partial_3 u_3^q})\end{aligned} \qquad (C_7^q)$$

General solution of the problem

From now on, we will suppose that $\rho^p = \rho^{p'}$ for any integer p, $p' < q$.

First layer: Assume that the displacement components are known functions $\underline{u}_j^1$ j = 1, 2, 3 on the free surface. From the equations (4), we deduce:

$$\begin{aligned}&\mu^1\underline{\partial_3 u_j^1} = -\mu^1\partial_j\underline{u}_3^1+\pi_j \qquad j=1,2 \\ &(\lambda^1+2\mu^1)\underline{\partial_3 u_3^1} = -(\lambda^1\partial_1\underline{u}_1^1+\lambda^1\partial_2\underline{u}_2^1)+\pi_3\end{aligned} \qquad (28)$$

and, applying the Fourier transformation to the two members of these equations:

$$\begin{aligned}&\mu^1\underline{\Phi}^1 = -\mu^1 d^2\underline{w}^1+\Psi \\ &(\lambda^1+2\mu^1)\underline{\varphi}^1 = -\lambda^1\underline{W}^1+\psi\end{aligned} \qquad (29)$$

where:

$$2i\pi\Psi = \xi_1\hat{\pi}_1+\xi_2\hat{\pi}_2, \quad 2i\pi\psi=\hat{\pi}_3 \quad (C_{10})$$

The equations (25)* and (26)*, considered in the first layer, determine $(\underline{W}^1, \underline{w}^1)$ and $(\underline{\Phi}^1, \underline{\varphi}^1)$ as functions of $(\underline{W}^1, \underline{w}^1)$ and (Ψ, ψ).

Second layer: Applying the Fourier transformation, the equations (7), (8), (9) become:

$$\begin{aligned}&\underline{W}^2 = \underline{W}^1, \quad \underline{w}^2 = \underline{w}^1 \\ &\mu^2\underline{\Phi}^2 = \mu^1\underline{\Phi}^1+(\mu^1-\mu^2)d^2\underline{w}^1 \\ &(\lambda^2+2\mu^2)\underline{\varphi}^2 = (\lambda^1+2\mu^1)\underline{\varphi}^1+(\lambda^1-\lambda^2)\underline{W}^1\end{aligned} \qquad (30)$$

From the equations (30), (25)* and (26)*, we get the equations (31)* and (32)*.

(p+1)-th layer: Assume $(\underline{W}^{p-1}, \underline{w}^{p-1})$ and (Φ^p, ϕ^p) computed; the equations (7), (8), (9) give, after the Fourier transformation:

$$\begin{aligned}&\underline{W}^{p+1} = \underline{W}^p, \quad \underline{w}^{p+1} = \underline{w}^p \\ &\mu^{p+1}\underline{\Phi}^{p+1} = \mu^p\underline{\Phi}^p+(\mu^p-\mu^{p+1})d^2\underline{w}^p \\ &(\lambda^{p+1}+2\mu^{p+1})\underline{\varphi}^{p+1} = (\lambda^p+2\mu^p)\underline{\varphi}^p+(\lambda^p-\lambda^{p+1})\underline{W}^p\end{aligned} \qquad (33)$$

so that (Φ^{p+1}, ϕ^{p+1}) are distributions given by (34)* and $(\underline{W}^p, \underline{w}^p)$ may be determined by (25)*.

q-th layer: The method just explained determines (W^q, w^q) and (Φ^q, ϕ^q) as functions of $(\underline{W}^1, \underline{w}^1)$ and (Ψ, ψ). Putting these values in (27)*, we obtain a system of linear equations from which $(\underline{W}^1, \underline{w}^1)$ are known as functions of (Ψ, ψ). Thus the Fourier transforms of the displacement components (and consequently of the stresses) are determined in every point by (17)*.

Systems with 4 layers: From the equations (31)*, (32)*, (25)* and (34)*, we get the equations (35)* and (36)* which give $(\underline{W}^3, \underline{w}^3)$ and $(\underline{\Phi}^3, \underline{\phi}^3)$ as functions of $(\underline{W}^1, \underline{w}^1)$ and (Ψ, ψ), in which we use the abbreviating symbols (C_{11})* and the symbols obtained from them by permutation of the letters in the following way: from the symbol S_{A1}, we deduce S_{A2} by permuting in the expression of the symbol S_{A1} the exponents 1 and 2; S_{B1}, by permuting A^1 and B^1 as well as A^2 and B^2.

In the 4-th layer, the equations (27)* give $(\underline{\Phi}^4, \underline{\phi}^4)$ as functions of $(\underline{W}^4, \underline{w}^4)$. From the equations (33), we deduce $(\underline{\Phi}^3, \underline{\phi}^3)$ as functions of $(\underline{W}^4, \underline{w}^4)$ given by the equations (37)*. Putting these values in the equations which give $(\underline{W}^3, \underline{w}^3)$ as functions of $(\underline{W}^3, \underline{w}^3)$, we get the system (38):

$$\begin{aligned}&2\rho r^2 a^3 b^3(A^4B^4-d^2)B^3\underline{W}^3 = L_A\underline{W}^4+id^2J_A\underline{w}^4 \\ &2\rho r^2 a^3 b^3(A^4B^4-d^2)A^3\underline{w}^3 = -iJ_B\underline{W}^4+L_B\underline{w}^4\end{aligned} \qquad (38)$$

in which J and L are given by (C_{12})*.

From the systems (37)* and (26)*, we deduce the system (39)*.

Writing that (35)* and (38)* give the same values as (36)* and (39)* for $(\underline{W}^3, \underline{w}^3)$ and $(\underline{\Phi}^3, \underline{\phi}^3)$, we obtain the system (40)* of 4 linear equations in which we use the following abbreviations:

$$\begin{aligned}&A^1X_{A1} = 2(\mu^3-\mu^2)(2\mu^1A^1G_{A1}+B^2D_{B2}) \\ &B^1Z_{A1} = 2(\mu^3-\mu^2)d^2(-2B^1H_{A1}+B^2F_{A2})\end{aligned} \qquad (C_{13})$$

This linear system may be solved by the ordinary Cramer method. The explicit formulas are rather complicated, but easily obtained. We omit them here because they are not of great ineterest for the purpose of this paper, but useful for numerical computations. They are of the following form:

$$\begin{aligned}&\Lambda\underline{W}^1 = i\,\Theta\,\Psi + d^2\,\Xi\,\psi \\ &\Lambda\underline{w}^1 = -\Xi'\Psi + i\,\Theta'\psi\end{aligned} \qquad (41)$$

where Λ, Θ, Ξ, Θ' and Ξ' are polynomials of the variables τ, ξ_1, ξ_2, of square roots of the polynomials $(A^j)^2$, $(B^j)^2$, j = 1, 2, 3, 4, and of functions $e^{-\pi n^p K^p}$, n^p denoting the thickness of the p-th layer and $K^p = A^p$ or B^p.

The formulas corresponding to the case of 4 layers can be applied for the case of less than 4 layers by identification of corresponding constants. In particular, we have the final equations (42)* in the case of one layer.

Final results:

From (41), we deduce:

$$\underline{u}_j^1 = \check{\mathcal{F}}_{\tau,\xi_1,\xi_2}\left[\frac{\Theta}{2\pi\Lambda}\hat{\pi}_j - i\xi_j\frac{\Xi}{2\pi\Lambda}\hat{\pi}_3\right] = \check{S}_j \quad , \quad j = 1, 2$$

$$\underline{u}_3^1 = \check{\mathcal{F}}_{\tau,\xi_1,\xi_2}\left[\frac{\Xi'}{2\pi\Lambda}\left(\xi_1\hat{\pi}_1 + \xi_2\hat{\pi}_2\right) + \frac{\Theta'}{2\pi\Lambda}\hat{\pi}_3\right] = \check{S}_3$$

The distributions u_j are in fact defined by functions. This may be proved directly by our method but is also a known result for hyperbolic systems as said before. If the boundary conditions are regular enough, we have:

$$\underline{u}_j^1 = \int_{-\infty}^{+\infty} e^{2i\pi t\tau}\,d\tau \iint e^{2i\pi(x_1\xi_1 + x_2\xi_2)}\,S_j(\tau,\xi_1,\xi_2)\,d\xi_1 d\xi_2$$

where the integrals are ordinary integrals which may be computed with an electronic computer. These regularity conditions are satisfied for example: 1) In the case of a plate in contact with the free surface and vibrating in any given direction; 2) In the case of heavy bodies moving along the free surface with constant velocity or constant acceleration.

For given values of the elastic parameters, it is possible to indicate simple approximations of the solutions which could be used for experimental purposes; such an approximation depends only on a certain range of values for the ratios of the parameters. In two cases, these approximations gave numerical results which were in agreement with experimental data (corresponding to small values of the Young modulus E^p, where p = 1, 2, 3) and explained some observed irregularities.

The older methods of harmonic analysis which have been applied to system (1) (for one or two layers) may give the solution only for very regular boundary conditions, whereas our method gives the solution in all cases which are of practical interest, since the boundary conditions must only be expressible in terms of temperate distributions.

The same method can be applied in the case of other boundary conditions and, more generally, to solve all mixed Cauchy problems with respect to hyperbolic systems.

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