T-complexes and crossed complexes

By N. Ashley[1]

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Introduction: This thesis is concerned with the development of the theory of *T-complexes* and their applications to topology.

Recently R. Brown and P.J. Higgins have been working on generalisations of the theorem of Seifert-van-Kampen to two dimensions [4] and to higher dimensions [3]. In their generalisation the space with base point of the Seifert-van-Kampen theorem is replaced by a filtered space $X : X_0 \subset X_1 \subset \cdots \subset X_n \subset \cdots$ and the fundamental group (or more generally groupoid) is replaced by the homotopy crossed complex $\pi_*(X)$ which in dimension $n \geq 2$ is the usual relative homotopy groups $\pi_n(X_n, X_{n-1}, a)$ ($a \in X_0$) and which has the usual boundary maps and operations of the fundamental groupoid. There is in fact a category C of *crossed complexes* which have the formal properties that $\pi_*(X)$ enjoys, and this category includes the category of chain complexes of modules over groups. We note that crossed complexes were originally defined by Whitehead [5] and called by him homotopy systems.

A crucial aspect of the work of [3] is to replace the crossed complex $\pi_*(X)$ by what the authors call an *ω-groupoid* $\rho(X)$, see [2], and a key result of their theory is that the categories of ω-groupoids and of crossed complexes are equivalent. The ω-groupoid $\rho(X)$ behaves better with regard to subdivision and concepts such as the homotopy addition lemma, than does $\pi_*(X)$.

In [2] ω-groupoids are defined as certain cubical sets with extra elements of structure. However because of the importance of simplicial theory for mathematics it has seemed desirable to have a simplicial version of ω-groupoids.

The key step for this was taken by M.K. Dakin [1] in defining a *T-complex*. T-complexes are special Kan complexes K with certain special elements in each dimension for $n \geq 1$. These special elements are called *thin* and satisfy the following three axioms:

T.1 Every degenerate element is thin.

T.2 Every box in K has a unique thin filler.

[1]Originally published by Esquisses Mathématiques, volume 32, in 1983.

T.3 A thin filler of a thin box has a thin shell.

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The three axioms for a T-complex were first stated by Dakin as an analysis of properties of the nerve K of a groupoid. For such a K every element of K_n is thin for $n > 1$.

Dakin [1] showed also how to associate a crossed complex to a T-complex.

In chapter 1 we prove the important result that the categories of T-complexes and crossed complexes are equivalent. The proof is an generalisation of the theory of Dold-Kan relating simplicial abelian groups and chain complexes. However our proof is considerably more complicated than that of the Dold-Kan theorem as among others it incorporates the relative homotopy addition lemma.

The theory developed in chapter 1 shows that T-complexes have a very rich algebraic structure and this is one reason for investigating such objects.

In chapter 2 we define the notion of a *special filtered Kan complex.* Given a filtered Kan complex $K : K_0 \subset K_1 \subset \cdots \subset K_n \subset \cdots$ we show how to associate a Kan complex $R(K)$ and a T-complex $\rho(K)$ to such an object.

We relate this theory to the work of R. Brown and P.J. Higgins [3]. They construct for a filtered space $X : X_0 \subset X_1 \subset \cdots \subset X_n \subset \cdots$ a cubical Kan complex $R(X)$ and by imposing a relation of filtered homotopy on $R(X)$ obtain a cubical T-complex, provided each loop in X_0 is contractible in X_1. We show in the simplicial context that the analogous $R(X)$ can be given the structure of a special filtered Kan complex K such that $R(K) = R(X)$ and $\rho(K) = \rho(X)$. This gives the important geometric example of a T-complex.

A consequence of our results is that, in analogy to the Brown-Higgins result, the crossed complex associated to the simplicial T-complex $\rho(X)$ is essentially the homotopy crossed complex $\pi_*(X)$.

In chapter 3 we apply our theory developed in chapters 1 and 2 to the category of simplicial groups. This category has found many applications in algebraic topology and other areas. This suggests the question of the relationship, if any, between T-complexes and simplicial groups.

In this chapter we define the notion of a *group T-complex* in the category of simplicial groups and obtain an explicit formula for determining when a simplicial group is a group T-complex. It then turns out that every simplicial abelian group is a group T-complex.

We next show that every T-complex contains a group T-complex and this leads us to define a new category whose objects are certain group T-complexes over a groupoid. We then show that this new category is equivalent to the category of T-complexes.

Finally in chapter 4 we prove some miscellaneous results about T-complexes.

Appendix: (reprint from page 76 to ensure legibility)
The proof of the uniqueness in theorem 9.1 of chapter 1.

Let $f, \bar{f} : (J, S) \to (K, T)$ be two morphisms of T-complexes agreeing on $N(J, S)$. Then we must show that $f = \bar{f}$. We use induction. We assume that $f_i = \bar{f}_i$ for $0 \leq i \leq n-1$. Then we must show that $f_n = \bar{f}_n$. By theorem 9.1 for $u \in J_{n-1}$ $f_n^u = \bar{f}_n^u$. But for $x \in J_n$, $s_{n-2} x_n x I[x_{n-1}] \in J_n^s n{-}2^d n{-}1^x n$ and so we

have that $f_n(s_{n-2}x_nxI[x_{n-1}]) = \bar{f}_n(s_{n-2}x_nxI[x_{n-1}])$. Then by 7.8(1) of chapter 1 it follows that $s_{n-2}f_{n-1}x_nf_nxI[f_{n-1}x_{n-1}] = s_{n-2}\bar{f}_{n-1}x_n\bar{f}_nxI[\bar{f}_{n-1}x_{n-1}]$. Now by our inductive hypothesis and lemma 8.4 we have that $f_nx = \bar{f}_nx$ as required.

T-COMPLEXES and CROSSED COMPLEXES

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DECLARATION

The work of this thesis has been carried out by the candidate and contains the results of his own investigations. The work has not already been accepted in substance for any degree, and is not being concurrently submitted in candidature for any degree. All sources of information have been acknowledged in the text.

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CONTENTS

CHAPTER 4

INTRODUCTION

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The theory developed in chapter 1 shows that T-complexes have a very rich algebraic structure and this is one reason for investigating such objects.

In chapter 2 we define the notion of a <u>special filtered Kan complex.</u> Given a filtered Kan complex $\underline{K} : K_0 \subset K_1 \subset \ldots \subset K_n \subset \ldots$ we show how to associate a Kan complex $R(\underline{K})$ and a T-complex $\rho(\underline{K})$ to such an object.

We relate this theory to the work of R. Brown and P.J. Higgins [3]. They construct for a filtered space $\underline{X} : X_0 \subset X_1 \subset \ldots \subset X_n \subset \ldots$ a cubical Kan complex $R(\underline{X})$ and by imposing a relation of filtered homotopy on $R(\underline{X})$ obtain a cubical T-complex , provided each loop in X_0 is contractible in X_1. We show in the simplicial context that the analogous $R(\underline{X})$ can be given the structure of a special filtered Kan complex $\underline{K}$ such that $R(\underline{K}) = R(\underline{X})$ and $\rho(\underline{K}) = \rho(\underline{X})$. This gives the important geometric example of a T-complex.

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In chapter 3 we apply our theory developed in chapters 1 and 2 to the category of simplicial groups. This category has found many applications in algebraic topology and other areas. This suggests the question of the relationship,if any , between T-complexes and simplicial groups.

In this chapter we define the notion of a group T-complex in the category of simplicial groups and obtain an explicit formula for determing when a simplicial group is a group T-complex. It then turns out that every simplicial abelian group is a group T-complex.

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PRELIMINARIES

Let $K = ((K_n)_{n\geq 0}, d_i, s_i)$ be a simplicial set. Recall that a simplicial set K is said to be a Kan complex if for every collection of n elements $x_0,\ldots,x_{k-1},x_{k+1},\ldots,x_n$ in K_{n-1} which satisfy the compatibility condition $d_i x_j = d_{j-1} x_i$ for $i<j$, $i\neq k$, $j\neq k$, (for the sake of brevity, we say that in this case, $(x_0,\ldots,x_{k-1},-,x_{k+1},\ldots,x_n)$ constitutes a box.) there exists an element $x\in K_n$ such that $d_i x = x_i$ $i\neq k$. For such an x we say that x is a filler of our box. By a shell in K is meant a collection of n+1 elements $x_0,\ldots,x_n$ in K_{n-1} which satisfy the compatibility condition $d_i x_j = d_{j-1} x_i$ $i<j$.

By $\Delta(n)$ is meant the free simplicial set with one generator δ^n of dimension n. By $\Delta(n,r)$ is meant the r skeleton of $\Delta(n)$. If λ is a proper subset of the set $\{0,\ldots,n\}$, then $\bar{\Lambda}^{\lambda}(n)$ denotes the subcomplex of $\Delta(n)$ generated by all $d_i\delta^n$ for $i\in\lambda$. For convenience we write $\Lambda^k(n)$ for $\bar{\Lambda}^{\{0,\ldots,k-1,k+1,\ldots,n\}}(n)$. Recall that $\Delta(n)\times\Delta(1)$ is generated by the elements $(s_i\delta^n, s_n\ldots s_{i+1}s_{i-1}\ldots s_0\delta^1)$ $i = 0,\ldots,n$.

By a groupoid is meant a sextuple (G,H,d_1,d_0,m,s) in which G,H are sets, $d_1,d_0 : G\longrightarrow H$ are respectively the initial and final maps, m is the partial composition on G and $s : H\longrightarrow G$ is the identity function; these data are to satisfy the usual axioms.

The following definition is due to Dakin [1]

Definition.

A T-complex consists of a pair (K,T) where K is a simplicial set and $T = (T_n)_{n\geq 1}$ is a graded subset of K with $T_n\subset K_n$. Elements of T are called thin and the following axioms are satisfied:

T.1 Every degenerate element is thin.

T.2 Every box in K has a unique thin filler.

T.3 A thin filler of a thin box has a thin shell.

Then T-complexes are the objects of a category T in which the morphisms are simplicial maps preserving thin elements.

By a T-complex of rank n is meant a T-complex (K,T) such that $K_m = T_m$ for $m>n$.

The following definition is due to Brown-Higgins [2] and is a generalisation of a definition of Whitehead [5].

<u>Definition</u>.

A <u>crossed complex</u> c is a sequence

$$\cdots \longrightarrow c_n \xrightarrow{\delta_n} c_{n-1} \longrightarrow \cdots \longrightarrow c_3 \xrightarrow{\delta_3} c_2 \xrightarrow{\delta_2} c_1 \underset{\delta_0}{\overset{\delta_1}{\rightrightarrows}} c_0$$

and satisfying the following axioms:

C.1 $(c_1, c_0, \delta_1, \delta_0)$ is a groupoid with objects in c_0, initial and final maps δ_1, δ_0 respectively. We write $c_1(a,b)$ for the collection of arrows from a to b $(a,b \varepsilon c_0)$ and $c_1(a)$ for the group $c_1(a,a)$.

C.2 For $n \geq 2$, c_n is a family of groups $(c_n(a))_{a \varepsilon c_0}$ and for $n \geq 3$ these groups are abelian.

C.3 The groupoid c_1 operates to the right on each c_n, $n \geq 2$, with an action $(x,p) \longmapsto x^p$ for $x \varepsilon c_n(a)$, $p \varepsilon c_1(a,b)$. Then $x^p \varepsilon c_n(b)$ and the usual laws hold.

C.4 For $n \geq 2$ $\delta_n : c_n \longrightarrow c_{n-1}$ is a morphism of groupoids over c_0 which preserves the action of c_1, where c_1 operates on the groups $c_1(a)$ by conjugation.

C.5 If $x \varepsilon c_2$ then $\delta_2 x$ operates trivially on c_n for $n \geq 3$ and operates on c_2 by conjugation by x.

C.6 $\delta_n \delta_{n+1}$ is trivial for $n \geq 2$.

We remark that these axioms imply that for $a \varepsilon c_0$, $(c_2(a), c_1(a), \delta_2)$ is a crossed module (originally defined by Whitehead in [5]).

Then crossed complexes are the objects of a category $\mathcal{C}$ in which a morphism $f : c \longrightarrow c'$ of crossed complexes is a family of maps $f_n : c_n \longrightarrow c'_n$ for $n \geq 0$ compatible with all the groupoid structures, the maps δ and the action of c_1 on c_i.

Notation.

Throughout the rest of this thesis we will adopt the following notation. Let $K = ((K_n)_{n \geq 0}, d_i, s_i)$ be a simplicial set. Then for $x \varepsilon K_n$ we write x_i for $d_i x$, $d_i^p x$ for $d_i \ldots d_i x$ and $s_i^p x$ for $s_i \ldots s_i x$.

We will sometimes refer to $d_i x$ or x_i as the i face of x and elements of the form $s_i x$ will be called degenerate.

As a standard source of reference we refer the reader to May, J.P. : Simplicial objects in algebraic topology. Princeton : Van Nostrand Co., Inc. 1967

CHAPTER 1

In this chapter we shall prove that the categories of T-complexes and crossed complexes are equivalent. To modify this problem consider the following.

Definition

A T-complex (K,T) is said to be connected if for every pair (a,b) in K_0 there exists an element $p \varepsilon K_1$ such that $p_0 = a$ and $p_1 = b$.

Proposition.

Every T-complex is the disjoint union of connected T-complexes.

Proof. Let (K,T) be a T-complex. Then we let $(K_0^\alpha)_{\alpha \varepsilon I}$ be the connected components of K_0, indexed by I. Since every box in K_1 has a filler, it follows that $K_0^\alpha \cap K_0^\beta$ is empty for $\alpha \neq \beta$ and so $K_0 = \bigcup_{\alpha \varepsilon I} K_0^\alpha$. We now define for $n \geq 1$ and $\alpha \varepsilon I$, $K_n^\alpha = \{x \varepsilon K_n : d_0^n x \varepsilon K_0^\alpha$ and for $n \geq 1$ $T_n^\alpha = K_n^\alpha \cap T_n$. Then it is clear that (K^α, T^α) is a well defined T-complex such that $(K^\alpha, T^\alpha) \cap (K^\beta, T^\beta)$ is empty for $\alpha \neq \beta$. Hence $(K,T) = \bigcup_{\alpha \varepsilon I} (K^\alpha, T^\alpha)$. This completes the proof.

It now follows that to prove the equivalence of the categories T and C it is sufficient to prove that the category of connected T-complexes is equivalent to the category of connected crossed complexes (crossed complexes whose groupoid structure in dimension one is connected.) Hence throughout the rest of this chapter, whenever we refer to a T-complex (crossed complex) we will mean a connected T-complex (crossed complex).

The proof that the categories T and C are equivalent will be divided up into twelve sections.

In section 1 we show that for a T-complex (K,T), $(K_n, K_{n-1}, d_n, d_{n-1})$ can be given a groupoid structure. We note that this result was proved by Dakin [1]. Then for $u \in K_{n-1}$ we let K_u^n be the group of arrows from u to itself.

In section 2 we construct an isomorphism $h_n[u,v] : K_u^n \longrightarrow K_v^n$ ($u,v \in K_{n-1}$).

In section 3 we define, for $n \geq 3$ and $a \in K_0$, abelian subgroups $K_a^n(A) \subset K^n_{s_0^{n-1}a}$

In section 4 we define homomorphisms $\delta_n : K^n_{s_0^{n-1}a} \longrightarrow K^{n-1}_{s_0^{n-2}a}$ such that $\delta_n \delta_{n+1}$ trivial. Thus we have constructed a sequence :

$$\cdots \longrightarrow (K_a^n(A))_{a\in K_0} \xrightarrow{\delta_n} \cdots \longrightarrow (K_a^3(A))_{a\in K_0} \xrightarrow{\delta_3} (K_a^2)_{a\in K_0} \xrightarrow{\delta_2} K_1 \underset{d_1}{\overset{d_0}{\rightrightarrows}} K_0$$

of groups and homomorphisms which we denote by $N(K,T)$.

In section 5 we define for $p,q \in K_1$ with $p_0 = q_1$ an isomorphism $\Phi : K^n_{s_0^{n-2}p} \longrightarrow K^n_{s_0^{n-2}pq}$. Then using Φ and the homomorphism h, defined in section 2, we define an action of K_1 on K_a^n for $a \in K_0$.

In section 6 we do the technical work which enables us to prove in section 7 that $N(K,T)$ is a crossed complex. We note that Dakin [1] has shown how to associate a crossed complex to a T-complex. We then define a functor $N : T \longrightarrow C$.

In section 8 we prove some technical results which will be used in sections 9 and 10.

In section 9 we show that a morphism $f : (J,S) \longrightarrow (K,T)$ of T-complexes is an isomorphism if and only if $f \mid N(J,S) \longrightarrow N(K,T)$ is an isomorphism of crossed complexes.

In section 10 we define the T-complex addition lemma which is analogous to the relative homotopy addition lemma.

In section 11 we use the T-complex addition lemma to define a functor

$D : C \longrightarrow T$.

Finally in section 12 we show that the functors D and N define an equivalence of categories. We note that Dakin [1] has proved that T-complexes of rank 2 are equivalent to crossed modules over groupoids.(The notion of a crossed module over a groupoid is due to R.Brown and P.J.Higgins [2]).

§1. A groupoid structure for T-complexes.

In this section we show how the axioms for a T-complex (K,T) enable one to define a canonical groupoid K^n ($n\geq 1$) with objects in K_{n-1} and arrows in K_n. We then show that for $i = 0,\ldots,n-2$ $d_i : K^n \longrightarrow K^{n-1}$ and $s_i : K^n \longrightarrow K^{n+1}$ are functors of groupoids. We note that Dakin [1] has shown that $(K_n, K_{n-1}, d_r, d_{r-1})$ is a groupoid for $r = 1,\ldots n$. For completeness and notational purposes we reprove his results for the case $(K_n, K_{n-1}, d_n, d_{n-1})$.

Throughout the rest of this section we let (K,T) denote a T-complex.

We begin by showing how to define a partial law of composition on K.

Lemma 1.1

There is a unique family $(M[x,y])_{n+1}$ of thin elements for $n\geq 1$ and $x,y\in K_n$ with $x_{n-1} = y_n$ such that :

$(M[x_0,y_0],\ldots M[x_{n-2},y_{n-2}],y,-,x)$ is a box and $M[x,y]\in T_{n+1}$ is its thin filler.

Proof. We use induction. Suppose that the lemma is true for dimension n and suppose that for all pairs $u,v\in K_{n-1}$ with $u_{n-2} = v_{n-1}$ the thin elements $M[u,v]$ have been assigned. Now let $x,y\in K_n$ with $x_{n-1} = y_n$. To prove the lemma we must check that the postulated faces of $M[x,y]$ are well defined and do actually fit together. Firstly $x_{n-1} = y_n$ implies $d_{n-2}x_i = d_{n-1}y_i$ for $0\leq i\leq n-2$.

Hence by our inductive hypothesis, $M[x_i, y_i]$ is defined. Thus all the postulated faces of $M[x,y]$ certainly do exist and we now check that they form a box. Denoting these faces by f_i $(i \neq n)$ we have to check that $d_i f_j = d_{j-1} f_i$ for all $0 \leq i < j \leq n+1$ and $i,j \neq n$. There are four cases.

(1) $0 \leq i < j \leq n-2$.

$d_i f_j = d_i M[x_j, y_j] = M[x_{j,i}, y_{j,i}] = M[x_{i,j-1}, y_{i,j-1}] = d_{j-1} f_i$.

(2) $0 \leq i < j = n-1$.

$d_i f_{n-1} = d_i y = d_{n-2} M[x_i, y_i] = d_{n-2} f_i$.

(3) $0 \leq i < j = n+1$ and $i \neq n-1$.

$d_i f_{n+1} = d_i x = d_n M[x_i, y_i] = d_n f_i$.

(4) $i = n-1$, $j = n+1$.

$d_{n-1} f_{n+1} = x_{n-1} = y_n = d_n f_{n-1}$.

Thus we have shown that the postulated faces do constitute a box and we take its thin filler to be $M[x,y]$.

Finally to start the induction if $n = 1$ and $p,q \varepsilon K_1$ with $p_0 = q_1$ we define $M[p,q] \varepsilon T_2$ to be the thin filler of the box $(q,-,p)$.

Using lemma 1.1 we now define a partial law of composition on K_n $(n \geq 1)$ as follows. Suppose $x,y \varepsilon K_n$ with $x_{n-1} = y_n$. Then we define $xy = d_n M[x,y]$. Notice that by virtue of the uniqueness of the thin element $M[x,y]$, the law of composition is both well defined and canonical.

<u>Proposition 1.2.</u>

Let $x,y \varepsilon K_n$ with $x_{n-1} = y_n$, then

(a) $(xy)_i = x_i y_i$ for $0 \leq i \leq n-2$.

(b) $(xy)_{n-1} = y_{n-1}$.

(c) $(xy)_n = x_n$.

(d) If x,y are both thin, then so to is xy whereas if $x \varepsilon T_n$, $y \notin T_n$ then $xy \notin T_n$.

Proof.

(a) For $0\leq i\leq n-2$, $(xy)_i = d_i d_n M[x,y] = d_{n-1}M[x_i,y_i] = x_i y_i$.

(b) $(xy)_{n-1} = d_{n-1}d_n M[x,y] = d_{n-1}d_{n-1}M[x,y] = y_{n-1}$.

(c) $(xy)_n = d_n d_n M[x,y] = d_n d_{n+1} M[x,y] = x_n$.

(d) Follows from axiom T.3 of a T-complex.

Theorem 1.3.

For $n\geq 1$, $K^n = (K_n, K_{n-1}, d_n, d_{n-1}, M, s_{n-1})$ is a groupoid.

Proof. We have to check :

(a) Associativity.

Lemma 1.4.

There is a unique family $(W[x,y,w])_{n+2}$ of thin elements for $n\geq 1$ and $x,y,w\in K_n$ with $x_{n-1} = y_n$, $y_{n-1} = w_n$ such that :

$(W[x_0,y_0,w_0],\ldots,W[x_{n-2},y_{n-2},w_{n-2}],M[y,w],-,M[x,yw],M[x,y])$ is a box and $W[x,y,w]\in T_{n+2}$ is its thin filler.

Proof. We use induction. Suppose that the lemma is true for dimension $n+1$ and suppose that for all triples $(r,u,v)\in K_{n-1}$ with $r_{n-2} = u_{n-1}$ and $u_{n-2} = v_{n-1}$ the thin elements $W[r,u,v]$ have been assigned. Now let $x,y,w\in K_n$ with $x_{n-1} = y_n$, $y_{n-1} = w_n$. Then to prove the lemma we must check that the postulated faces of $W[x,y,w]$ are well defined and do actually fit together. Firstly $x_{n-1} = y_n$ and $y_{n-1} = w_n$ imply that $x_{i,n-2} = y_{i,n-1}$ and $y_{i,n-2} = w_{i,n-1}$ for $0\leq i\leq n-2$. Hence, by our inductive hypothesis, $W[x_i,y_i,w_i]$ is defined. Also by 1.2(c) $M[x,yw]$ is well defined. Thus all the postulated faces of $W[x,y,w]$ certainly do exist and we now check that they form a box. Denoting these faces by f_i $(i\neq n)$ we have to show that $d_i f_j = d_{j-1} f_i$ for all $0\leq i<j\leq n+2$ and $i,j\neq n$. There are five cases.

(1) $0\leq i<j\leq n-2$.

$d_i f_j = d_i W[x_j, y_j, w_j] = W[x_{i,j-1}, y_{i,j-1}, w_{i,j-1}] = d_{j-1} f_i.$

(2) $0 \le i < j = n-1$.

$d_i f_{n-1} = d_i M[y,w] = M[y_i, w_i] = d_{n-2} W[x_i, y_i, w_i] = d_{n-2} f_i.$

(3) $0 \le i < j = n+1$, $i \ne n-1$.

$d_i f_{n+1} = d_i M[x, yw] = M[x_i, y_i w_i] = d_n f_i.$

(4) $i = n-1$, $j = n+1$.

$d_{n-1} f_{n+1} = d_{n-1} M[x, yw] = yw = d_n M[y,w] = d_n f_{n-1}.$

(5) $0 \le i < j = n+2$.

$$d_i f_{n+2} = d_i M[x,y] = \left\{ \begin{array}{l} M[x_i, y_i] = d_{n+1} f_i \ , \ 0 \le i \le n-2. \\ y = d_{n+1} M[y,w] = d_{n+1} f_{n-1}, \ i = n-1. \\ x = d_{n+1} M[x,yw] = d_{n+1} f_{n+1}, \ i = n+1. \end{array} \right\}$$

Thus we have shown that the postulated faces do constitute a box and we take its thin filler to be $W[x,y,w]$.

Finally to start the induction if $n = 1$ and $p,q,s \in K_1$ with $p_0 = q_1$ and $q_0 = s_1$ we define $W[p,q,s] \in T_3$ to be the thin filler of the box $(M[q,s], -, M[p,qs], M[p,q])$.

Lemma 1.5.

Let $x,y,w \in K_n$ with $x_{n-1} = y_n$, $y_{n-1} = w_n$. Then $d_n W[x,y,w] = M[xy,w]$.

Proof. We use induction. Suppose that the lemma is true for dimension $n+1$, that is for every triple $(r,u,v) \in K_{n-1}$ with $r_{n-2} = u_{n-1}$, $u_{n-2} = v_{n-1}$ we have that $d_{n-1} W[r,u,v] = M[ru,v]$. Now let $x,y,w \in K_n$ with $x_{n-1} = y_n$ and $y_{n-1} = w_n$. Then $d_i d_n W[x,y,w] = d_{n-1} W[x_i, y_i, w_i] = M[x_i y_i, w_i]$ $0 \le i \le n-2$, $d_{n-1} d_n W[x,y,w] = d_{n-1} d_{n-1} W[x,y,w] = w = d_{n-1} M[xy,w]$, and $d_{n+1} d_n W[x,y,w] = d_n d_{n+2} W[x,y,w] = xy = d_{n+1} M[xy,w]$. Hence $d_n W[x,y,w]$ and $M[xy,w]$ fill the same box. But by axiom T.3 of a T-complex, $d_n W[x,y,w] \in T_{n+1}$ and so by axiom T.2 we have that $d_n W[x,y,w] = M[xy,w]$.

Finally it is clear that the lemma is true for the case $n = 1$.

Corollary 1.6.

$(xy)w = x(yw)$ when defined.

(b) Identities. and $x = s_{n-1}x_n x$

Let $x \varepsilon K_n$. Then we need to show that $x s_{n-1} x_{n-1} = x$ and $x = s_{n-1}x_n x$. It is sufficient to prove :

Lemma 1.7.

Let $x \varepsilon K_n$. Then (a) $M[x, s_{n-1}x_{n-1}] = s_n x$ for $n \geq 1$. (b) $M[s_{n-1}x_n, x] = s_{n-1}x$

Proof (a) We use induction. We suppose that the lemma is true for dimension n, that is for all u K_{n-1}, $M[u, s_{n-2}u_{n-2}] = s_{n-1}u$. Now let $x \varepsilon K_n$, then

$d_i M[x, s_{n-1}x_{n-1}] = M[x_i, s_{n-2}x_{i,n-2}] = s_{n-1}x_i$ for $0 \leq i \leq n-2$,

$d_{n-1}M[x, s_{n-1}x_{n-1}] = s_{n-1}x_{n-1} = d_{n-1}s_n x$ and

$d_{n+1}M[x, s_{n-1}x_{n-1}] = x = d_{n+1}s_n x$. Hence $M[x, s_{n-1}x_{n-1}]$ and $s_n x$ both fill the same box and so by axiom T.2 are equal.

Finally it is clear that the lemma is true for the case n = 1.
(b) Similiar to (a)

(c) Existence of Inverses.

Lemma 1.8

There is a unique family $(I[x])_{n+1}$ of thin elements for $n \geq 1$ and $x \varepsilon K_n$ such that :

$(I[x_0], \ldots, I[x_{n-2}], -, s_{n-1}x_n, x)$ is a box and $I[x] \varepsilon T_{n+1}$ is its thin filler. The proof is similiar to 1.1 and 1.4 and so will be omitted.

We now define the inverse of $x \varepsilon K_n$ to be $d_{n-1}I[x]$ which we denote by x^{-1}. Then it is clear that x^{-1} has as its shell $(x_0^{-1}, \ldots, x_{n-2}^{-1}, x_n, x_{n-1})$ and so xx^{-1} and $x^{-1}x$ are well defined. To show that $xx^{-1} = s_{n-1}x_n$ it is sufficient to prove :

Lemma 1.9.

If $x \varepsilon K_n$ $(n \geq 1)$, then $M[x, d_{n-1}I[x]] = I[x]$.

Proof. We use induction. Wesuppose the lemma is true in dimension n, that

is for every $u\varepsilon K_{n-1}$) $I[u] = M[u,d_{n-2}I[u]]$. Now let $x\varepsilon K_n$, then $d_iM[x,d_{n-1}I[x]] = M[x_i,d_{n-2}I[x_i]] = I[x_i]$ $0\le i\le n-2$, $d_{n-1}M[x,d_{n-1}I[x]] = d_{n-1}I[x]$ and $d_{n+1}M[x,d_{n-1}I[x]] = x = d_{n+1}I[x]$. Hence $I[x]$ and $M[x,d_{n-1}I[x]]$ fill the same box and so by axiom T.2 are equal.

Finally it is clear that the lemma is true for the case $n = 1$.

This completes the proof of theorem 1.3.

We write $K^n(u,v)$ for the collection of arrows from u to v ($u,v\varepsilon K_{n-1}$) and K^n_u for the group $K^n(u,u)$.

<u>Proposition 1.10.</u>

For $0\le i\le n-2$ we have that :

(a) $d_i : K^n \longrightarrow K^{n-1}$ is a functor of groupoids.

(b) $s_i : K^n \longrightarrow K^{n+1}$ is a functor of groupoids.

<u>Proof.</u>

(a) Use 1.2(a).

(b) Let $x,y\varepsilon K_n$ with $x_{n-1} = y_n$. Then for $0\le i\le n-2$ $d_ns_ix = s_id_{n-1}x = s_id_ny = d_{n+1}s_iy$, and so s_ixs_iy is well defined. Further by axiom T.1 and 1.2(d) we have that s_ixs_iy and s_ixy are both thin. Now by a simple inductive argument it follows that s_ixs_iy and s_ixy both have the same shell and so are equal.

In general K^n ($n\ge 2$) is not a connected groupoid as can be seen from the following :

<u>Proposition 1.11.</u>

If $u,v\varepsilon K_{n-2}$ ($n\ge 2$) with $u\ne v$, then $K^n(s_{n-2}u,s_{n-2}v)$ is empty.

<u>Proof.</u> Let $(u,v)\varepsilon K_{n-2}$ with $u\ne v$. Now suppose there exists an element $x\varepsilon K^n(s_{n-2}u,s_{n-2}v)$. Then $d^2_{n-1}x = d_{n-1}d_nx$ implying $u = v$, hence contradiction.

§2. The isomorphism theorem for a T-complex.

Let (K,T) be a T-complex. Recall that in section one we proved that $K^n = (K_n, K_{n-1}, d_n, d_{n-1}, M, s_{n-1})$ is a groupoid. Hence for $u \varepsilon K_{n-1}$ $K^n_u = \{x \varepsilon K_n : x_n = x_{n-1} = u\}$ can be given a group structure. We then proved that in general K^n is not a connected groupoid. However we prove the following.

Theorem 2.1. (The isomorphism theorem)

Let (K,T) be a T-complex. Then for $n \geq 1$ and every pair $(u,v) \varepsilon K_{n-1}$ with $d_0^{n-1}u = d_0^{n-1}v$ there exists a canonical isomorphism $h[u,v] : K^n_u \longrightarrow K^n_v$ and satisfying :

(1) $h[u,u]$ is the identity.

(2) $h[u,v]$ maps thin elements to thin elements.

(3) If $u,v,w \varepsilon K_{n-1}$ with $d_0^{n-1}u = d_0^{n-1}v = d_0^{n-1}w$, then $h[v,w] \circ h[u,v] = h[u,w]$.

(4) For all pairs $(u,v) \varepsilon K_{n-1}$ with $d_0^{n-1}u = d_0^{n-1}v$ and for $i = 0,\ldots,n-2$ we have that $d_i h[u,v] = h[u_i, v_i] d_i : K^n_u \longrightarrow K^{n-1}_{v_i}$.

Note that since $d_0^{n-1}u = d_0^{n-1}v$ we have that $d_0^{n-2}u_i = d_0^{n-2}v_i$ $i = 0,\ldots,n-2$ and so $h[u_i, v_i]$ is well defined. We remark that for $a,b \varepsilon K_0$, $h[a,b]$ is defined only if $a = b$.

The proof of theorem 2.1 will occupy the rest of this section.

Lemma 2.2.

Let (K,T) be a T-complex. Then there is a unique family $(F[x,v])_{n+1}$ of thin elements for $n \geq 2$ and every pair x,v with $v \varepsilon K_{n-1}$ and $x \varepsilon K^n_v$ such that $(-, F[x_1,v_1], \ldots, F[x_{n-2}, v_{n-2}], x, s_{n-1}v, s_{n-1}v)$ is a box and $F[x,v] \varepsilon T_{n+1}$ is its thin filler.

Proof. We use induction. Suppose the lemma is true for dimension n and suppose that for all pairs y,u with $u \varepsilon K_{n-2}$ and $y \varepsilon K^{n-1}_u$ the thin elements

$F[y,u]\varepsilon T_n$ have been assigned. Now let $v\varepsilon K_{n-1}$ and $x\varepsilon K^n_v$. To prove the lemma we must check that the postulated faces of $F[x,v]$ are well defined and do actually fit together. Firstly $x\varepsilon K^n_v$ implies, by 1.10, that $x_i\varepsilon K^{n-1}_{v_i}$ for $i = 0,\ldots n-2$ and so by our inductive hypothesis $F[x_i,v_i]$ is defined. Thus all the postulated faces of $F[x,v]$ certainly do exist and we now check that they form a box. Denoting these faces by f_i $(i\neq 0)$ we have to check that $d_i f_j = d_{j-1}f_i$ for all $1\leq i<j\leq n+1$. There are four cases.

(1) $1\leq i<j\leq n-2$.

$$d_i f_j = d_i F[x_j,v_j] = F[x_{j,i},v_{j,i}] = F[x_{i,j-1},v_{i,j-1}] = d_{j-1}f_i.$$

(2) $1\leq i<j = n-1$.

$$d_i f_j = x_i = d_{n-2}F[x_i,v_i] = d_{n-2}f_i.$$

(3) $1\leq i<j = n$.

$$d_i f_n = d_i s_{n-1}v = \left\{\begin{array}{ll} s_{n-2}v_i = d_{n-1}F[x_i,v_i] = d_{n-1}f_i, & i\neq n-1. \\ \\ v = d_{n-1}x = d_{n-1}f_{n-1}, & i = n-1. \end{array}\right\}$$

(4) $1\leq i<j = n+1$.

This case is similiar to (3) and so will be omitted.

Finally to start the induction for $n = 2$, if $p\varepsilon K_1$ and $x\varepsilon K^2_p$ then we let $F[x,p]$ be the thin filler of the box $(-,x,s_1p,s_1p)$.

Corollary 2.3

Let $v\varepsilon K_{n-1}$ and $x\varepsilon K^n_v$. Then $d_0F[x,v]\varepsilon K^n_{s_{n-2}d_0v}$.

Lemma 2.4.

Let $v\varepsilon K_{n-1}$ and $x,\bar{x}\varepsilon K^n_v$, then

(1) $F[x,v]F[\bar{x},v] = F[x\bar{x},v]$.

(2) If $d_0F[x,v] = d_0F[\bar{x},v]$, then $x = \bar{x}$.

Proof.

(1) We use induction. We suppose the lemma is true in dimension $n-1$. That

is for all $u \varepsilon K_{n-2}$ and $y,\bar{y} \varepsilon K_u^{n-1}$, $F[y,u]F[\bar{y},u] = F[y\bar{y},u]$. Now let $v \varepsilon K_{n-1}$ and $x,\bar{x} \varepsilon K_v^n$. Then by 1.2(d) $F[x,v]F[\bar{x},v] \varepsilon T_{n+1}$. Hence to show that $F[x,v]F[\bar{x},v] = F[x\bar{x},v]$ it is sufficient, by axiom T.2 of a T-complex, to show that $d_i(F[x,v]F[\bar{x},v]) = d_i F[x\bar{x},v]$ $i = 1,\ldots,n+1$. But this follows from 1.2(a) and our inductive hypothesis.

(2) This is proved easily using induction.

We now define a function $f[v] : K_v^n \longrightarrow K^n_{s_{n-2}d_0 v}$ which is well defined

$$x \longmapsto d_0 F[x,v]$$

by 2.3. Furthermore f[v] is injective by 2.4(1) and 2.4(2) and since d_0 is a morphism (by 1.2) it is also a homomorphism.

<u>Lemma 2.5.</u>

Let (K,T) be a T-complex. Then there is a unique family $(G[y,v])_{n+1}$ of thin elements for $n \geq 2$ and every pair y,v with $v \varepsilon K_{n-1}$ and $y \varepsilon K^n_{s_{n-2}d_0 v}$ such that $(y, G[y_0,v_1],\ldots,G[y_{n-3},v_{n-2}],-,s_{n-1}v,s_{n-1}v)$ is a box and $G[y,v] \varepsilon T_{n+1}$ is its thin filler.

The proof is similiar to the proof of 2.2 and so will be omitted.

<u>Corollary 2.6.</u>

Let $v \varepsilon K_{n-1}$ and $y \varepsilon K^n_{s_{n-2}d_0 v}$, then $d_{n-1}G[y,v] \varepsilon K_v^n$.

<u>Lemma 2.7</u>

Let $v \varepsilon K_{n-1}$ and $y,\bar{y} \varepsilon K^n_{s_{n-2}d_0 v}$, then

(1) $G[y,v]G[\bar{y},v] = G[y\bar{y},v]$.

(2) If $d_{n-1}G[y,v] = d_{n-1}G[\bar{y},v]$, then $y = \bar{y}$.

The proof is similiar to the proof of 2.4 and so will be omitted.

We now define a function $g[v] : K^n_{s_{n-2}d_0 v} \longrightarrow K_v^n$ which is well

$$y \longmapsto d_{n-1}G[y,v]$$

defined by 2.6. Furthermore $g[v]$ is injective by 2.7(1) and 2.7(2) and since d_{n-1} is a morphism (by 1.2) it is also a homomorphism.

Lemma 2.8.

Let $v\varepsilon K_{n-1}$ and let $x\varepsilon K_v^n$, $y\varepsilon K^n_{s_{n-2}d_0v}$, then

(1) $G[d_0F[x,v],v] = F[x,v]$.

(2) $G[y,v] = F[d_{n-1}G[y,v],v]$.

Proof.

(1) We use induction. We suppose the lemma is true in dimension n, that is for all $u\varepsilon K_{n-2}$ and $w\varepsilon K_u^{n-1}$, $G[d_0F[w,u],u] = F[w,u]$. Now let $v\varepsilon K_{n-1}$ and $x\varepsilon K_v^n$. Then by our inductive hypothesis $G[d_0F[x,v],v]$ and $F[x,v]$ fill the same box and so by axiom T.2 of a T-complex they are equal.

Finally to start the induction for $n = 2$ if $p\varepsilon K_1$ and $x\varepsilon K_p^2$ then $G[d_0F[x,p],p]$ and $F[x,p]$ fill the box $(d_0F[x,p],-,s_1p,s_1p)$ and so are equal.

(2) The proof is similiar to (1) and so will be omitted.

Corollary 2.9.

Let $v\varepsilon K_{n-1}$. Then $f[v] : K_v^n \longrightarrow K^n_{s_{n-2}d_0v}$ is an isomorphism whose inverse is $g[v]$.

Notation.

For $n\geq 1$ let $v\varepsilon K_{n-1}$. Then we define $v^0 = v$ and for $i\geq 1$ $v^i = s_{n-2}d_0v^{i-1}$. Then it is clear that $v^j = s_0^{n-1}d_0^{n-1}v$ for $j\geq n-1$.

Let $v\varepsilon K_{n-1}$. Then we define

$\hat{f}[v] = f[v^{n-2}]\ldots f[v^1]f[v]$ and

$\hat{g}[v] = g[v]\ldots g[v^{n-3}]g[v^{n-2}]$.

Then for all pairs $(u,v)\varepsilon K_{n-1}$ with $d_0^{n-1}u = d_0^{n-1}v$ we define

$h[u,v] = \hat{g}[v]\hat{f}[u] : K^n_u \longrightarrow K^n_v$.

It is now straightforward to show that $h[u,v]$ satisfies the first three properties of theorem 2.1 and so we will prove property 4: that is we must show that for $0 \le i \le n-2$ and $u, v \varepsilon K_{n-1}$ with $d_0^{n-1}u = d_0^{n-1}v$ we have a commutative diagram

$$\begin{tikzcd} K^n_u \arrow[r, "{h[u,v]}"] \arrow[d, "d_i"'] & K^n_v \arrow[d, "d_i"] \\ K^{n-1}_{u_i} \arrow[r, "{h[u_i,v_i]}"'] & K^{n-1}_{v_i} \end{tikzcd}$$

We restrict ourselves to showing that for $v \varepsilon K_{n-1}$ and $x \varepsilon K^n_v n-2$ we have that $d_i\hat{g}[v]x = \hat{g}[v_i]x_i$ for $0 \le i \le n-2$.

Recall that $\hat{g}[v] = g[v]g[v^1]\dots g[v^{n-2}]$.

For $v \varepsilon K_{n-1}$ and $x \varepsilon K^n_v n-2$ let $g[v^1]g[v^2]\dots g[v^{n-2}]x = y$. Then

$$d_i g[v]y = d_i d_{n-1} G[y,v]$$

$$= d_{n-2} d_i G[y,v] = \begin{cases} d_{n-2}y & \text{for } i = 0 \\ g[v_i]d_{i-1}y & \text{for } 0 \le i \le n-2 \end{cases}$$

It now follows that $d_{n-2}g[v]\dots g[v^{n-2}]x = g[v_{n-2}]\dots g[v^{n-3}_{n-2}]d_0 g[v^{n-2}]x$ and

for $0 \le i \le n-3$ $d_i g[v]\dots g[v^{n-2}]x = g[v_i]\dots g[(d_i v)^{i-1}]d_{n-2}(g[v^{i+1}]\dots g[v^{n-2}]x)$. Thus it is sufficient to prove the following lemma.

<u>Lemma A.</u>

For $1 \le k \le n-3$ let $v \varepsilon K_{n-1}$ and $x \varepsilon K^n_v n-2$, then

$$d_{n-2}g[v^k]g[v^{k+1}]\dots g[v^{n-2}](x) = g[(d_{k-1}v)^{k-1}]g[(d_{k-1}v)^k]\dots g[(d_{k-1}v)^{n-3}]d_{k-1}x.$$

Proof.

Let $y = g[v^{k+1}]\ldots g[v^{n-2}](x)$. Then

$$
\begin{aligned}
d_{n-2}g[v^k]y &= d_{n-2}d_{n-1}G[y,v^k] \\
&= d_{n-2}d_{n-2}G[y,v^k] \\
&= d_{n-2}G[y_{n-3},d_{n-2}(v^k)] \text{ by 2.5} \\
&= g[d_{n-2}(v^k)]d_{n-3}y \\
&= g[(d_{k-1}v)^{k-1}]d_{n-3}y \quad \text{since } d_{n-2}v^k = d_{n-2}s_{n-2}d_0v^{k-1} = d_0v^{k-1} \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad = (d_{k-1}v)^{k-1}
\end{aligned}
$$

Now letting $w = g[v^{k+2}]\ldots g[v^{n-2}](x)$ we have that

$$
\begin{aligned}
d_{n-3}y = d_{n-3}g[v^{k+1}](w) &= d_{n-3}d_{n-1}G[w,v^{k+1}] \\
&= d_{n-2}d_{n-3}G[w,v^{k+1}] \\
&= d_{n-2}G[d_{n-4}w,d_{n-3}(v^{k+1})] \text{ by 2.5} \\
&= g[d_{n-3}(v^{k+1})]d_{n-4}w \\
&= g[(d_{k-1}v)^k]d_{n-4}w.
\end{aligned}
$$

We now suppose that for $1\le j<(n-1-k)$

$$d_{n-2}g[v^k]\ldots g[v^{n-2}]x = g[(d_{k-1}v)^{k-1}]\ldots g[(d_{k-1}v)^{k+j-2}]d_{n-2-j}g[v^{k+j}]\ldots g[v^{n-2}]x$$

Then letting $z = g[v^{k+j+1}]\ldots g[v^{n-2}](x)$ we have that

$$
\begin{aligned}
d_{n-2-j}g[v^{k+j}]z &= d_{n-2-j}d_{n-1}G[z,v^{k+j}] \\
&= d_{n-2}d_{n-2-j}G[z,v^{k+j}] \\
&= d_{n-2}G[d_{n-3-j}z,d_{n-2-j}(v^{k+j})] \\
&= g[d_{n-2-j}(v^{k+j})]d_{n-3-j}z \\
&= g[(d_{k-1}v)^{k+j-1}]d_{n-3-j}z.
\end{aligned}
$$

This completes the proof of lemma A. It now follows that for $v\varepsilon K_{n-1}$ and $x\varepsilon K_v^n{}_{n-2}$, $d_i\hat{g}[v](x) = \hat{g}[v_i](x_i)$ for $0\le i\le n-2$ as required.

§3. Certain abelian groups associated to a T-complex.

Let (K,T) be a T-complex. Recall that for $a\varepsilon K_0$ we have defined groups $K^n_{s_0^{n-1}a} = \{x\varepsilon K_n : x_n = x_{n-1} = s_0^{n-1}a\}$. Throughout the rest of chapter 1 where no confusion arises we shall write a for $s_0^{n-1}a$ whenever $a\varepsilon K_0$.

For $a\varepsilon K_0$ we define the following subsets of K^n_a.

$$K^n_a(A) = \begin{cases} \{x\varepsilon K^2_a : x_0 = s_0 a\} & \\ \{x\varepsilon K^n_a : x_i = s_0^{n-1}a \text{ for } i = 1,\ldots,n\} & n\geq 3 \end{cases}$$

$$K^n_a(B) = \begin{cases} \{x\varepsilon K^2_a : x_0 = s_0 a\} & \\ \{x\varepsilon K^n_a : x_i \varepsilon K^{n-1}_a(B) \text{ for } i = 0,\ldots,n-2\} & n\geq 3 \end{cases}$$

$T^n_a = K^n_a \cap T_n$ for $n\geq 2$.

Then it is clear that $K^n_a(A), K^n_a(B)$ and T^n_a are subgroups of K^n_a. We state the following lemma , the proof being clear from the definitions.

Lemma 3.1.

(a) $K^n_a(A) \subset K^n_a(B)$ for $n\geq 2$ and further $K^2_a(A) = K^2_a(B)$.

(b) $d_i : K^n_a(B) \longrightarrow K^{n-1}_a(B)$ is a homomorphism with $d_i K^n_a(A) \subset K^{n-1}_a(A)$ for $0\leq i\leq n$ and $n\geq 3$.

We are now ready to state the three main theorems of this section.

Theorem 3.2.

$K_a^n(A)$ is in the centre of K_a^n for $n\geq 2$.

Theorem 3.3.

K_a^n is isomorphic to $K_a^n(A) \times T_a^n$ for $n\geq 3$.

Theorem 3.4.

$K_a^n(B)$ is in the centre of K_a^n for $n\geq 2$.

The proofs will occupy the rest of section 3. We remark here that we use theorem 3.2 to prove theorem 3.4.

The proof of 3.2 consists of six lemmas, the proofs of which are omitted as they are similiar to those found in section one.

Lemma.

There is a unique family $\{m(x,\alpha)\}_{n+1}$ of thin elements for $n\geq 2$ and every pair $(x,\alpha)\varepsilon K_a^n$ with $\alpha\varepsilon K_a^n(A)$ such that :

(a) $(m(x_0,\alpha_0), s_{n-3}x_1, \ldots, s_{n-3}x_{n-3}, x, -, \alpha, s_0^n a)$ is a box and $m(x,\alpha)\varepsilon T_{n+1}$ is its thin filler.

(b) $m(x, s_0^n a) = s_{n-2}x$.

Lemma.

There is a unique family $\{J(x,\alpha)\}_{n+2}$ of thin elements for $n\geq 2$ and every pair $(x,\alpha)\varepsilon K_a^n$ with $\alpha\varepsilon K_a^n(A)$ such that :

(a) $(J(x_0,\alpha_0), M[s_{n-3}x_1, a], \ldots, M[s_{n-3}x_{n-3}, a], M[x,a], M[x,\alpha], s_{n-1}\alpha, -, s_{n-2}x)$ is a box and $J(x,\alpha)\varepsilon T_{n+2}$ is its thin filler.

(b) $J(x,a) = s_{n-2}M[x,a]$.

Lemma.

Let $(x,\alpha)\epsilon K_a^n$ with $\alpha\epsilon K_a^n(A)$, then $d_{n+1}J(x,\alpha) = m(x,\alpha)$.

Corollary.

Let $(x,\alpha)\epsilon K_a^n$ with $\alpha\epsilon K_a^n(A)$, then $x\alpha = d_{n-1}m(x,\alpha)$.

Lemma.

There is a unique family $\{R(x,\alpha)\}_{n+2}$ of thin elements for $n\geq 2$ and every pair $(x,\alpha)\epsilon K_a^n$ with $\alpha\epsilon K_a^n(A)$ such that :

(a) $(R(x_0,\alpha_0), s_{n-3}s_{n-3}x_1, \ldots, s_{n-3}s_{n-3}x_{n-3}, s_{n-2}x, s_{n-2}x, -, s_n\alpha, s_n\alpha)$ is a box and $R(x,\alpha)\epsilon T_{n+2}$ is its thin filler.

(b) $R(x,a) = s_{n-2}s_{n-2}x$.

Lemma.

There is a unique family $\{L(x,\alpha)\}_{n+2}$ of thin elements for $n\geq 2$ and every pair $(x,\alpha)\epsilon K_a^n$ with $\alpha\epsilon K_a^n(A)$ such that :

(a) $(L(x_0,\alpha_0), M[a,s_{n-3}x_1], \ldots, M[a,s_{n-3}x_{n-3}], M[a,x], M[\alpha,x], d_nR(x,\alpha), -, s_{n-1}\alpha)$ is a box and $L(x,\alpha)\epsilon T_{n+2}$ is its thin filler.

(b) $L(x,a) = s_{n-2}s_{n-1}x$.

Lemma.

Let $(x,\alpha)\epsilon K_a^n$ with $\alpha\epsilon K_a^n(A)$, then $d_{n+1}L(x,\alpha) = m(x,\alpha)$.

Corollary.

Let $(x,\alpha)\epsilon K_a^n$ with $\alpha\epsilon K_a^n(A)$. Then we have that $\alpha x = x\alpha$.

This completes the proof of theorem 3.2.

Proof of theorem 3.3.

Using theorem 3.2 and the fact that $K_a^n(A) \cap T_a^n = s_0^n a$ it is sufficient to show that $K_a^n(A)$ and T_a^n generate K_a^n. This we now do. Let $x\epsilon K_a^n$. Then we let $\bar{x}\epsilon T_a^n$ fill the box $(-, x_1^{-1}, \ldots, x_{n-2}^{-1}, s_0^{n-1}a, s_0^{n-1}a)$. Note that this box is well

defined since x^{-1} has as its shell $(x_0^{-1}, x_1^{-1}, \ldots, x_{n-2}^{-1}, s_0^{n-1}a, s_0^{n-1}a)$. Then by 1.2(a) $x\bar{x} \varepsilon K_a^n(A)$ and so we have that $x = (x\bar{x})\bar{x}^{(-1)}$. This completes the proof of theorem 3.3.

Proof of theorem 3.4.

We use induction and the two previous theorems. Suppose the theorem is true for $K_a^i(B)$ $i = 2,\ldots,n-1$. Note that it is true for $i = 2$ since $K_a^2(B) = K_a^2(A)$. Now let $x \varepsilon K_a^n(B)$. Then by theorem 3.3, x can be uniquely expressed as $x = \alpha y$ where $\alpha \varepsilon K_a^n(A)$ and $y \varepsilon T_a^n$. Furthermore it is clear that $y \varepsilon K_a^n(B)$. Now by 3.2 it is sufficient to show that y is in the centre of T_a^n. But by lemma 3.1 $y_i \varepsilon K_a^{n-1}(B)$ $i = 0,\ldots,n$ and so by our inductive hypothesis y_i is in the centre of K_a^{n-1}. Hence for any $w \varepsilon T_a^n$, yw and wy have the same shell and so are equal. This completes the proof of theorem 3.4.

§4. The homomorphism δ.

Let (K,T) be a T-complex. Then for $n \geq 2$ and $a \varepsilon K_0$ we define a function

$$\delta_n : K_a^n \longrightarrow K_a^{n-1} \qquad \text{which is clearly well defined.}$$
$$x \longrightarrow \prod_{i=0}^{n-2} x_i^{(-1)^i}$$

Theorem 4.1.

(1) δ_n is a homomorphism.

(2) $\delta_{n-1}\delta_n$ is trivial for $n \geq 3$.

(3) $\delta_n K_a^n(A) \subset K_a^{n-1}(A)$ for $n \geq 3$.

(4) $\delta_3 K_a^3 \subset K_a^2(A)$.

(5) $\delta_n T_a^n \subset T_a^{n-1}$ for $n \geq 3$.

(6) If $x \varepsilon T_a^n$, then $x_0 x_1^{-1} \ldots x_{n-2}^{(-1)^{n-2}}$ is a product of degenerate elements which depend only on the elements $x_{i,j}$.

(7) Letting $p_n : K_a^n \longrightarrow K_a^n(A)$ be the epimorphism whose kernel is T_a^n we have
$$x \longmapsto x\bar{x}$$
that $\delta_n p_n = p_{n-1}\delta_n$.

In order to prove theorem 4.1 we need the following lemma.

Lemma 4.2.

If $y \varepsilon K_a^n$ and $n \geq 3$, then $y_i y_{i+1}^{-1} \varepsilon K_a^{n-1}(B)$ for $0 \leq i \leq n-3$.

Proof. We use induction. Suppose the lemma is true for all elements in K_a^{n-1}. Now consider $y \varepsilon K_a^n$. Then to prove the lemma we need to show that $d_j(y_i y_{i+1}^{-1}) \varepsilon K_a^{n-2}(B)$ for $0 \leq j \leq n-3$ and $0 \leq i \leq n-3$. But

$$d_j(y_i y_{i+1}^{-1}) = \begin{cases} d_{i-1}y_j d_i y_j^{-1} & \text{for } i>j. \\ s_0^{n-2}a & \text{for } i = j. \\ d_i y_{j+1} d_{i+1} y_{j+1}^{-1} & \text{for } i<j. \end{cases}$$

and so by our inductive hypothesis $y_i y_{i+1}^{-1} \varepsilon K_a^{n-1}(B)$ for $0 \leq i \leq n-3$.

Finally to start the induction, if $y \varepsilon K_a^2$, then $d_0 y_0 = d_0 y_1$ and so $y_0 y_1^{-1} \varepsilon K_a^2(B)$. This completes the proof of lemma 4.2.

We are now ready to prove theorem 4.1.

(1) If $x, y \varepsilon K_a^n$, then $\delta_n x \delta_n y = x_0 x_1^{-1} x_2 \ldots x_{n-2}^{(-1)^{n-2}} y_0 y_1^{-1} y_2 \ldots y_{n-2}^{(-1)^{n-2}}$. Now applying lemma 4.2 and theorem 3.4 to the right hand side of the above equation we obtain $\delta_n x \delta_n y = \delta_n(xy)$ as required.

(2) If $x \varepsilon K_a^n$, then using (1) we have that $\delta_{n-1}\delta_n x = \prod_{i=0}^{n-2}\left(\prod_{j=0}^{n-3} x_{i,j}^{(-1)^{i+j}}\right)$ for $n \geq 3$. But $x_{i,j} = x_{j,i-1}$ for $i>j$ and so by applying 4.2 and 3.4 to the right hand side of the above equation we obtain $\delta_{n-1}\delta_n x = s_0^{n-2}a$ as required.

(3) If $y \varepsilon K_a^n(A)$, then $\delta_n y = d_0 y$. But for $0 \leq i \leq n-1$ $d_i d_0 y = d_0 d_{i+1} y = s_0^{n-2}a$. Hence $d_0 y \varepsilon K_a^{n-1}(A)$ as required.

(4) If $x\varepsilon K^3_a$, then $\delta_3 x = x_0 x_1^{-1} \varepsilon K^2_a(A)$ by 4.2.

(5) For $x\varepsilon T^n_a$ we let $x^{n-2} = x s_{n-3} x_{n-2}^{-1} \varepsilon T^n_a$ and for $i = n-3,\dots,1$ we let $x^i = x^{i+1} s_{i-1}(d_i x^{i+1})^{-1} \varepsilon T^n_a$. Then by a simple calculation we see that $d_i x^1 = s_0^{n-1} a$ for $1\le i\le n$. Hence $x^1 \varepsilon K^n_a(A)$, but $x^1 \varepsilon T^n_a$ and so $d_0 x = s_0^{n-1} a$. Now repeatedly substuting for x^i $i = 1,\dots,n-2$ and using 4.2 and 3.4 we obtain the required result. Alternatively the proof follows from the results in chapte 3

(6) This follows immediately from the proof of (5).

(7) If $x\varepsilon K^n_a$, then $\delta_n p_n x = \delta_n(x\bar{x}) = \delta_n x \delta_n \bar{x}$ by (1), whereas $p_{n-1}\delta_n x = \delta_n x(\overline{\delta_n x})$. Thus it is sufficient to show that $\delta_n \bar{x} = (\overline{\delta_n x})$. Recall that if $x\varepsilon K^n_a$, then $\bar{x}\varepsilon T^n_a$ fills the box $(-, x_1^{-1}, \dots, x_{n-2}^{-1}, s_0^{n-1}a, s_0^{n-1}a)$. Then

$$\delta_n \bar{x} = d_0\bar{x}\, d_1\bar{x}^{(-1)} \dots d_{n-2}\bar{x}^{(-1)^{n-2}} = d_0\bar{x}\, d_1 x \dots d_{n-2} x^{(-1)^{n-1}}$$

and so for $1\le i<n$ $d_i\delta_n\bar{x} = d_i d_0 \bar{x}\, d_i x_1 \dots d_i x_{n-2}^{(-1)^{n-1}} = d_i(\overline{\delta_n x})$. Hence $\delta_n \bar{x}$ and $(\overline{\delta_n x})$ both fill the same box and so by (5) and axiom T.2 of a T-complex we have that $\delta_n \bar{x} = (\overline{\delta_n x})$ as required.

This completes the proof of theorem 4.1.

Summary.

For a T-complex (K,T) we have constructed a sequence

$$\dots \longrightarrow (K^n_a(A))_{a\varepsilon K_0} \xrightarrow{\delta_n} \dots \longrightarrow (K^3_a(A))_{a\varepsilon K_0} \xrightarrow{\delta_3} (K^2_a)_{a\varepsilon K_0} \xrightarrow{\delta_2} K_1 \underset{d_0}{\overset{d_1}{\rightrightarrows}} K_0$$

and satisfying :

(1) $(K_1, K_0, d_1, d_0, M, s_0)$ is a groupoid.

(2) For $n\ge 3$ $(K^n_a(A))_{a\varepsilon K_0}$ is a family of abelian groups and $(K^2_a)_{a\varepsilon K_0}$ is a family of groups.

(3) δ_n is a morphism of groupoids over K_0 and $\delta_n\delta_{n+1}$ is trivial for $n\geq 2$.

We denote this sequence by $N(K,T)$. The next three sections will be concerned with showing that $N(K,T)$ can be given the structure of a crossed complex (see definition in preliminaries).

§5. The groupoid action.

Let (K,T) be a T-complex. In this section we show that the groupoid K^1 operates to the right on each $(K^n_a)_{a\varepsilon K_0}$ $(n\geq 2)$ with an action $(x,p)\longmapsto x^p \varepsilon K^n_{p_0}$ for $p\varepsilon K_1$ and $x\varepsilon K^n_{p_1}$.

The method used draws its motivation from the following well known geometrical case.

Let X be a topological space with a base point x. Let $W[X,x] = \{f : (S^1,*)\longrightarrow(X,x)\}$ denote the space of loops with $d_x : S^1 \longrightarrow x$ the constant loop. Then it is well known, see Hu.[6], that we have a bijection $\pi_n(X,x)\longrightarrow \pi_{n-1}(W[X,x],d_x)$ for $n\geq 1$. Now consider a path $p : I\longrightarrow X$ with $p(i) = x_i$, $i = 0,1$. Then p induces maps

(a) $\Phi : W[X,x_1]\longrightarrow W[X,x_0]$

$$f \longmapsto p^{-1}fp$$

(b) $H : I\longrightarrow W[X,x_0]$ such that $H(0) = d_{x_0}$, $H(1) = \Phi d_{x_1}$.

Thus we can form an isomorphism $p_n : \pi_n(X,x_1)\longrightarrow \pi_n(X,x_0)$ as follows :

$$\begin{tikzcd}
\pi_n(X,x_1) \arrow[rr, "p_n"] \arrow[d, leftrightarrow] & & \pi_n(X,x_0) \arrow[d, leftrightarrow] \\
\pi_{n-1}(W[X,x_1],d_{x_1}) \arrow[r, "\Phi^*"] & \pi_{n-1}(W[X,x_0],\Phi d_{x_1}) \arrow[r, "H"] & \pi_{n-1}(W[X,x_0],d_{x_0})
\end{tikzcd}$$

Correspondingly for $p\varepsilon K_1$ we will define an isomorphism

$K^n_{p_1} \xrightarrow{\phi} K^n_{s_0^{n-2}p} \xrightarrow{h} K^n_{p_0}$ where h is defined as in theorem 2.1 and ϕ is

defined as follows.

Theorem 5.1.

For every pair $(p,q)\varepsilon K_1$ with $p_0 = q_1$ there exists a canonical isomorphism $\phi_n[p,pq] : K^n_{s_0^{n-2}p} \longrightarrow K^n_{s_0^{n-2}pq}$ for $n\geq 2$ and $\phi_1[q] : K^1_{q_1} \longrightarrow K^1_{q_0}$ and satisfying the following properties :

(1) $\phi_n[p,p]$ is the identity for $n\geq 2$.

(2) $\phi_1[q] : K^1_{q_1} \longrightarrow K^1_{q_0}$

$$y \longmapsto q^{-1}yq$$

(3) If $p,q,r\varepsilon K_1$ with $p_0 = q_1$, $q_0 = r_1$, then $\phi_n[pq,pqr]\phi_n[p,pq] = \phi_n[p,pqr]$.

(4) $\phi_n[p,q]$ maps thin elements to thin elements.

(5) For $0\leq i\leq n-2$ and all pairs $(p,q)\varepsilon K_1$ with $p_0 = q_1$ we have that for $n\geq 3$ $\phi[p,pq]d_i = d_i\phi[p,pq] : K^n_{s_0^{n-2}p} \longrightarrow K^{n-1}_{s_0^{n-3}pq}$ and for $n = 2$ we have that $\phi_1[q]d_0 = d_0\phi_2[p,pq] : K^2_p \longrightarrow K^1_{q_0}$.

Proof. For every pair $(p,q)\varepsilon K_1$ with $p_0 = q_1$ and $x\varepsilon K^n_{s_0^{n-2}p}$ $(n\geq 2)$ we define $\phi_n[p,pq]x = (s_0^{n-2}M[p,q])^{-1}x\, s_0^{n-2}M[p,q]$ where $M[p,q]$ is defined as in section 1. Then it is clear that $\phi_n[p,pq]x$ is a well defined element of $K^n_{s_0^{n-2}pq}$. Then with this definition it is easy to check that $\phi_n[p,pq]$ is an isomorphism and satisfies the properties of theorem 5.1.

We now define an action of K^1 on $(K^n_a)_{a\varepsilon K_0}$ $(n\geq 2)$ by $x^p = h_n[s_0^{n-2}p, s_0^{n-1}p_0]\phi_n[s_0p_1,p]x$ whenever $p\varepsilon K_1$ and $x\varepsilon K^n_{p_1}$.

§6. The interchange law between the isomorphisms h and ϕ.

Let (K,T) be a T-complex. Where no confusion arises we shall write p for $s_0^m p$ whenever $p\varepsilon K_1$.

This section will be concerned with proving the following theorem.

Theorem 6.1.

Let $p,q,r\varepsilon K_1$ with $p_0 = q_0 = r_1$. Then for $n\geq 2$ we have that $h[pr,qr]\phi[p,pr] = \phi[q,qr]h[p,q]$.

The proof will occupy the rest of section 6.

Recall that for $u\varepsilon K_{n-1}$ we define $u^0 = u$ and for $i\geq 1$ $u^i = s_{n-2}d_0u^{i-1}$. Then $u^j = s_0^{n-1}d_0^{n-1}u$ for $j\geq n-1$. Furthermore we have that for $p\varepsilon K_1$ $s_{n-1}(s_0^{n-2}p)^j = (s_0^{n-1}p)^{j+1}$. Hence for $p\varepsilon K_1$ and $x\varepsilon K_p^n j$ implies $F[x,p^j]\varepsilon K_p^{n+1}j+1$, where $F[x,p^j]\varepsilon T_{n+1}$ is as defined in 2.4.

To enable us to prove theorem 6.1 we need the following two technical results.

Lemma 6.2.

Suppose $p,q\varepsilon K_1$ with $p_0 = q_0$ and $x\varepsilon K_p^n j$, then we have that for $n\geq 2$ $h_{n+1}[p^{j+1},q^{j+1}]F[x,p^j] = F[h_n[p^j,q^j](x),q^j]$.

Proof. We use induction. Suppose the lemma is true for dimension n, that is for all $v\varepsilon K_p^{n-1}j$ we have that $h_n[p^{i+1},q^{i+1}]F[v,p^i] = F[h_{n-1}[p^i,q^i](v),q^i]$. Now consider $x\varepsilon K_p^n j$. Then by 2.1(4) and our inductive hypothesis we have that $d_ih[p^{j+1},q^{j+1}]F[x,p^j] = d_iF[h[p^j,q^j](x),q^j]$ for $1\leq i\leq n+1$. Hence by 2.1(3) we have the required result.

Finally to start the induction for $n = 2$ let $x\varepsilon K_p^2 j$. Then $h[p^{j+1},q^{j+1}]F[x,p^j]$ and $F[h[p^j,q^j](x),q^j]$ both fill the box $(-,h[p^j,q^j](x),q^{j+1},q^{j+1})$ and so are equal. This completes the proof.

For every pair $p,r\varepsilon K_1$ with $p_0 = r_1$ we define $R[p,r;j] = s_0^{n-1-j}s_1^jM[p,r]$ for $0\leq j\leq n-1$, $n\geq 2$ and $R[p,r;j] = R[p,r;n-1]$ for $j\geq n-1$, $n\geq 2$.

For every pair $p,r\varepsilon K_1$ with $p_0 = r_1$ and every $x\varepsilon K_p^n j$ we define for $j\geq 0$ and

$n\geq 2$ thin elements $S[x,p,r;j+1] = R[p,r;j+1]^{-1}F[x,p^j]R[p,r;j+1]\varepsilon K^{n+1}_{R[p,r;j]}$.
Then for every triple $(p,q,r)\varepsilon K_1$ with $p_0 = q_0 = r_1$ and every $x\varepsilon K^n_p j$ we define for $j\geq 0$ and $n\geq 2$ $C[x,p,q,r;j+1] = h[R[p,r;j],R[q,r;j]]S[x,p,r;j+1]\varepsilon T_{n+1}$.

Lemma 6.3.

Suppose $p,q,r\varepsilon K_1$ with $p_0 = q_0 = r_1$ and $x\varepsilon K^n_p j$; then we have that $C[x,p,q,r;j+1] = S[h[p^j,q^j](x),q,r;j+1]$ for $j\geq 0$ and $n\geq 2$.

Proof. We use double induction. Suppose the theorem is true in dimension n, that is for all $v\varepsilon K^{n-1}_p i$ we have that $C[v,p,q,r;i+1] = S[h[p^i,q^i](v),q,r;i+1]$.
Now consider $x\varepsilon K^n_p n-1$. We first show that $C[x,p,q,r;n] = S[h[p^{n-1},q^{n-1}]x,q,r;n$
$= S[x,q,r;n]$ by 2.1(1).

By our inductive hypothesis and axiom T.2 of a T-complex it is sufficient to show that $d_0C[x,p,q,r;n] = d_0S[x,q,r;n]$. But

$$\begin{aligned}d_0C[x,p,q,r;n] &= h[r,r]d_0S[x,p,r;n]\\ &= d_0S[x,p,r;n] \text{ by } 2.1(1)\\ &= (s_0^{n-1}r)^{-1}d_0F[x,s_0^{n-1}p_0]s_0^{n-1}r\\ &= (s_0^{n-1}r)^{-1}d_0F[x,s_0^{n-1}q]\ s_0^{n-1}r\\ &= d_0S[x,q,r;n].\end{aligned}$$

We now suppose that for $j<i\leq n-1$ with $x\varepsilon K^n_p i$ we have that $C[x,p,q,r;i+1] = S[h[p^i,q^i](x),q,r;i+1]$. We now prove the lemma is true when $x\varepsilon K^n_p j$. By our first inductive hypothesis it is sufficient to show that $d_0C[x,p,q,r;j+1] = d_0S[h[p^j,q^j](x),q,r;j+1]$. But

$$\begin{aligned}d_0C[x,p,q,r;j+1] &= d_{n-1}C[d_0F[x,p^j],p,q,r;j+2] \text{ and}\\ d_0S[h[p^j,q^j](x),q,r;j+1] &= d_{n-1}S[d_0F[h[p^j,q^j](x),q^j],q,r;j+2].\\ &= d_{n-1}S[d_0h[p^{j+1},q^{j+1}]F[x,p^j],q,r;j+2] \text{ by } 6.2.\end{aligned}$$

Hence by our second inductive hypothesis we have the required result.

Finally to start the induction for $n = 2$ it is sufficient to show that for $x \varepsilon K_p^2$, $d_0C[x,p,q,r;1] = d_0S[h\ p,q\ (x),q,r;1]$. But

$$
\begin{aligned}
d_0C[x,p,q,r;1] &= h[r,r]d_0S[x,p,r;1] \\
&= (s_0r)^{-1}d_0F[x,p]s_0r. \quad \text{and} \\
d_0S[h[p,q](x),q,r;1] &= (s_0r)^{-1}d_0F[h[p,q](x),q]s_0r \\
&= (s_0r)^{-1}d_0h[p^1,q^1]F[x,p]s_0r \quad \text{by 6.2} \\
&= (s_0r)^{-1}d_0F[x,p]s_0r \quad \text{as required.}
\end{aligned}
$$

This completes the proof of lemma 6.3.

We are now ready to prove theorem 6.1. Suppose $p,q,r \varepsilon K_1$ with $p_0 = q_0 = r_1$ and $x \varepsilon K_p^n$. Then by 6.3 $d_{n-1}C[x,p,q,r;1] = d_{n-1}S[h\ p,q\ (x),q,r;1]$ and so $h[pr,qr]\phi[p,pr](x) = \phi[q,qr]h[p,q](x)$. This completes the proof of theorem 6.1.

§7. The crossed complex associated to a T-complex.

In this section we show how to associate a crossed complex to a T-complex. We note that Dakin [1]has also shown how to associate a crossed complex to a T- complex but using a different method than the one used here.

Recall that for a T-complex (K,T) we have constructed a sequence $N(K,T)$

$$
\cdots \longrightarrow (K_a^n(A))_{a\varepsilon K_0} \xrightarrow{\delta_n} \cdots \longrightarrow (K_a^3(A))_{a\varepsilon K_0} \xrightarrow{\delta_3} (K_a^2)_{a\varepsilon K_0} \xrightarrow{\delta_2} K_1 \underset{d_0}{\overset{d_1}{\rightrightarrows}} K_0
$$

Theorem 7.1.

Let (K,T) be a T-complex. Then $N(K,T)$ is a crossed complex.

Proof. By earlier results it suffic es to prove that $N(K,T)$ satisfies the axioms C.3, C.4 and C.5 for a crossed complex as stated in the preliminaries. Recall that for $p \varepsilon K_1$ and $x \varepsilon K_p^n$ $(n \geq 2)$ we have defined an action of K_1 on $(K_a^n)_{a\varepsilon K_0}$

by $x^p = h_n[s_0^{n-2}p, s_0^{n-1}p_0]\phi_n[s_0p_1,p](x) \varepsilon K^n_{p_0}$ (see section 5)

$= \phi_n[p^{-1}, s_0p_0]h_n[s_0^{n-1}p_1, s_0^{n-2}p^{-1}](x)$ by 6.1.

<u>Lemma 7.2.</u>

For $n \geq 2$ let $p,q \varepsilon K_1$ with $p_0 = q_1$ and suppose $x,y \varepsilon K^n_p$; then

(a) $(x^p)^q = x^{pq}$

(b) $(xy)^p = x^py^p$

<u>Proof.</u>

(a) $(x^p)^q = (h[s_0^{n-2}p, s_0^{n-1}p_0]\phi[s_0p_1,p](x))^q$

$= \phi[q^{-1}, s_0q_0]h[s_0^{n-1}q_1, s_0^{n-2}q^{-1}]h[s_0^{n-2}p, s_0^{n-1}p_0]\phi[s_0p_1,p](x)$

$= \phi[q^{-1}, s_0q_0]h[s_0^{n-2}p, s_0^{n-2}q^{-1}]\phi[s_0p_1,p](x)$ by 2.1(3)

$= h[pqs_0q_0, s_0q_0]\phi[p,pq]\phi[s_0p_1,p](x)$ by 6.1

$= h[pqs_0q_0, s_0q_0]\phi[s_0p_1,pq](x)$ by 6.1

$= h[pq, s_0q_0]\phi[s_0p_1,pq](x)$ by 1.7

$= x^{pq}$.

(b) Follows from the fact that h and ϕ are isomorphisms, see 2.1 and 5.1.

<u>Lemma 7.3.</u>

Suppose $p \varepsilon K_1$ and $x \varepsilon K^n_{p_1}$, then for $n \geq 3$ and $0 \leq i \leq n$ we have that $d_ix^p = (x_i)^p$.

Proof. For $0 \leq i \leq n-2$ $d_ix^p = d_ih[s_0^{n-2}p, s_0^{n-1}p_0]\phi[s_0p_1,p](x)$

$= h[s_0^{n-3}p, s_0^{n-2}p_0]\phi[s_0p_1,p](x_i)$ by 2.1(4) and 5.1(5)

$= x_i^p$

Lastly by 7.2(b) $(s_0^{n-1}p_1)^p = s_0^{n-1}p_0$.

<u>Corollary 7.4.</u>

Suppose $p \varepsilon K_1$ and $x \varepsilon K^n_{p_1}(A)$; then $x^p \varepsilon K^n_{p_0}(A)$.

Thus we have shown that N(K,T) satisfies axiom C.3.

Lemma 7.5.

Suppose $a\varepsilon K_0$, then for $n\geq 2$ $\delta_n : K_a^n \longrightarrow K_a^{n-1}$ preserves the action of K_1 where K_1 operates on the groups K_a^1 by conjugation.

Proof. For $p\varepsilon K_1$ and $x\varepsilon K_p^n$ we need to show that $\delta_n(x^p) = (\delta_n x)^p$. But for $n\geq 3$

$$\delta_n(X^p) = \prod_{i=0}^{n-2} (d_i x^p)^{(-1)^i} = \prod_{i=0}^{n-2} (d_i x^{(-1)^i})^p \quad \text{by 7.3}$$
$$= (\prod_{i=0}^{n-2} d_i x^{(-1)^i})^p \quad \text{by 7.2(b)}$$
$$= (\delta_n x)^p,$$

while for $n = 2$ $\delta_2(x^p) = d_0 x^p = d_0(h[p,s_0p_0]\phi[s_0p_1,p](x))$

$$= h[p_0,p_0]\phi_1[p](x_0)$$
$$= \phi_1[p](x_0) = p^{-1}x_0 p \quad \text{by 5.1(2)}.$$

Thus we have shown that $N(K,T)$ satisfies axiom C.4. It remains to show that $N(K,T)$ satisfies axiom C.5.

For $n\geq 2$ and $a\varepsilon K_0$ we define a function $e_n(a) : K_a^n \longrightarrow K_a^1$, $x \longmapsto d_0^{n-1}x$. Then it is clear , by 1.2(a) that $e_n(a)$ is a well defined homomorphism.

Lemma 7.6.

For $a\varepsilon K_0$ and $n\geq 2$, $e_n(a) : K_a^n \longrightarrow K_a^1$ is a crossed module.

Before proving 7.6 we need to say what is a crossed module.

A crossed module (originally defined by Whitehead [5]) is a triple (A,B,f) where $f : A \longrightarrow B$ is a morphism of groups together with ~~such that there is~~ a group action of B on A $(a,b)\mapsto a^b$ written a^b and satisfying (1) $f(a^b) = b^{-1}f(a)b$ (2) $a^{f(a_1)} = a_1^{-1}aa_1$ for $a,a_1\varepsilon A$ and $b\varepsilon B$.

Proof of 7.6.

By earlier results it suffices to show that $x^{e_n(a)y} = y^{-1}xy$ for $x,y\varepsilon K_a^n$. For $n\geq 2$ and $a\varepsilon K_0$, let $x\varepsilon K_a^n$ and $u\varepsilon K_a^i$ where $2\leq i\leq n$, with $e_i(a)u = p$. Then we define thin elements

$$T[x,u;i] = \begin{cases} (s_{i-1}^{n-i}M[s_0^i a,u])^{-1}s_{i-2}x s_{i-1}^{n-i}M[s_0^i a,u] & \text{for } 2\leq i\leq n-1 \\ M[s_0^n a,u]^{-1}s_{n-2}xM[s_0^n a,u] & \text{for } i = n \end{cases}$$

where $M[s_0^i a,u]$ is defined as in section 1.We need the following technical result.

Lemma 7.7.

$h_{n+1}[s_{i-1}^{n-i}u,s_0^n a]T[x,u;i] = (s_{i-2}x)^p$ for $2\leq i\leq n-1$ and

$h_{n+1}[u,s_0^n a]T[x,u;n] = (s_{n-2}x)^p$ for $i = n$.

Proof. We use double induction. Suppose the lemma is true in dimension n-1; that is for all $y\varepsilon K_a^{n-1}$ and $v\varepsilon K_a^j$ where $2\leq j\leq n-1$ with $e_i(a)v = q$ we have that

$h_n[s_{j-1}^{n-1-j}v,s_0^{n-1}a]T[y,v;j] = (s_{j-2}y)^q$ for $2\leq j\leq n-2$ and

$h_n[v,s_0^{n-1}a]T[y,v;n-1] = (s_{n-3}y)^q$ for $j = n-1$.

Now let $x\varepsilon K_a^n$ and $u\varepsilon K_a^2$. Then by our inductive hypothesis for $2\leq i\leq n+1$

$d_i h_{n+1}[s_1^{n-2}u,s_0^n a]T[x,u;2] = d_i(s_0 x)^p = (d_i s_0 x)^p$ by 7.3. Further

$d_0 h_{n+1}[s_1^{n-2}u,s_0^n a]T[x,u;2] = h_n[s_0^{n-2}d_0 u,s_0^{n-1}a]\phi[s_0 a,d_0 u](x) = x^{d_0 u} = x^p$.

Hence $h_{n+1}[s_1^{n-2}u,s_0^n a]T[x,u;2]$ and $(s_0 x)^p$ both fill the same box and so by 2.1(2) and axiom T.2 of a T-complex they are equal.

We now suppose that for $x\varepsilon K_a^n, u\varepsilon K_a^j$ and for $2\leq j<n$

$h_{n+1}[s_{j-1}^{n-j}u,s_0^n a]T[x,u;j] = (s_{j-2}x)^p$. Then we must show that for $x\varepsilon K_a^n$ and $u\varepsilon K_a^{j+1}$ $h_{n+1}[s_j^{n-(j+1)}u,s_0^n a]T[x,u;j+1] = (s_{j-1}x)^p$. But by our first inductive hypothesis for $k\neq j-1,j$ $d_k h_{n+1}[s_j^{n-(j+1)}u,s_0^n a]T[x,u;j+1] = d_k(s_{j-1}x)^p$. Further by our second inductive hypothesis

$d_{j-1}h_{n+1}[s_j^{n-(j+1)}u,s_0^n a]T[x,u;j+1] = d_{j-1}h_{n+1}[s_{j-1}^{n-j}u,s_0^n a]T[x,u;j] = d_{j-1}(s_{j-2}x)^p$.

Hence $h_{n+1}[s_j^{n-(j+1)}u, s_0^n a]T[x,u;j+1]$ and $(s_{j-1}x)^p$ both fill the same box and so by 2.1(2) and axiom T.2 of a T-complex they are equal. This completes the proof of lemma 7.7.

Now let $x,y\varepsilon K_a^n$. Then by lemma 7.7 we have that
$d_{n-1}h_{n+1}[y,s_0^n a]T[x,y;n] = d_{n-1}(s_{n-2}x)^{e_n(a)y}$ and so by 2.1(4) and 7.3 we have that $y^{-1}xy = x^{e_n(a)y}$ as required. This completes the proof of 7.6.

Now suppose $x\varepsilon K_a^n(A)$ $(n\geq 3)$ and $y\varepsilon K_a^2$ with $\delta_2 y = d_0 y = p\varepsilon K_1$. Then $s_0^{n-2}y\varepsilon K_a^n$ and $e_n(a)s_0^{n-2}y = p$. Hence

$$x^{\delta_2 y} = x^{e_n(a)s_0^{n-2}y} = (s_0^{n-2}y)^{-1}xs_0^{n-2}y \quad \text{by 7.6}$$

$$= x \quad \text{by 3.2}$$

Thus the image of δ_2 acts trivially on $K_a^n(A)$ $(n\geq 3)$ .Further for $x\varepsilon K_a^2$ and $y\varepsilon K_a^2$, $x^{\delta_2 y} = x^{d_0 y} = x^{e_2(a)y} = y^{-1}xy$ by 7.6. This completes the proof of theorem 7.1.

We end this section by defining a functor from T-complexes to crossed complexes.

Let $f : (J,S) \longrightarrow (K,T)$ be a morphism of T-complexes . Then since f preserves thin elements the following proposition is evident using induction.

Proposition 7.8.

(1) $f_{n+1}M[x,y] = M[f_n x, f_n y]$ whenever $x,y\varepsilon J_n$ with $x_{n-1} = y_n$ and $M[x,y]$ is defined as in section 1.

(2) $f_{n+1}I[x] = I[f_n x]$ for $x\varepsilon J_n$ and $I[x]$ is defined as in 1.8.

(3) If $v\varepsilon J_{n-1}$, then $f_n^v : J_v^n \longrightarrow K_{f_{n-1}v}^n$ is a morphism of groups.

$$x \longmapsto f_n x$$

(4) Let $u,v\varepsilon J_{n-1}$ with $d_0^{n-1}u = d_0^{n-1}v$. Then $f_n^v h[u,v] = h[f_{n-1}u, f_{n-1}v]f_n^u$, where $h[u,v]$ is defined as in section 2.

(5) Let $p,q\varepsilon K_1$ with $p_0 = q_1$. Then writing p for $s_0^{n-2}p$ we have that $f_n^{pq}\phi[p,pq] = \phi[f_1p,f_1pf_1q]f_n^p : J_p^n \longrightarrow K_{f_1(pq)}^n$ where $\phi[p,pq]$ is defined as in section 5.

Now using 7.8 it is clear that if $f : (J,S) \longrightarrow (K,T)$ is a morphism of T-complexes then f restricted to $N(J,S)$ is a morphism of crossed complexes. Thus we have a well defined functor $N : T \longrightarrow C$.

§8. Some technical results.

In this section we establish some technical results , notably lemma 8.5, which will be used in sections nine and ten.

Let (K,T) be a T-complex. Recall that for $x\varepsilon K_n$ we define $I[x]\varepsilon T_{n+1}$ to be the thin filler of the box $(I[x_0],\ldots,I[x_{n-2}],-,s_{n-1}x_n,x)$ and we define $x^{-1} = d_{n-1}I[x]$. (see section 1) Then x^{-1} has as its shell $(x_0^{-1},\ldots,x_{n-2}^{-1},x_n,x_{n-1})$.

Lemma 8.1.

Let (K,T) be a T-complex and suppose $x\varepsilon K_n$; then

(a) $I[x]s_{n-1}x = s_nx$ and so $I[x] = (s_{n-1}x)^{-1}$.

(b) $s_{n-1}xI[x] = s_ns_{n-1}x_n$.

(c) $s_nx = (s_nx)^{-1}$.

This is proved easily using induction and so will be omitted.

Let (K,T) be a T-complex. For $n\geq 2$ we define
$$k_n : K_n \longrightarrow K_n$$
$$x \longmapsto (s_{n-2}x_n)xI[x_{n-1}]$$

and for $v\varepsilon K_{n-1}$ we define
$$k_n^v : K_v^n \longrightarrow K_n .$$
$$x \longmapsto k_nx$$

Lemma 8.2.

If $x\varepsilon K_n$; then

$$d_i k_n x = \begin{cases} k_{n-1}x_i & \text{for } 0\le i\le n-3 \text{ and } n\ge 3 \\ x_n x_{n-2} x_{n-1}^{-1} & i = n-2 \\ s_{n-2}d_{n-1}x_n & \text{for } i = n-1, n. \end{cases}$$

Proposition 8.3.

(1) k_n maps thin elements to thin elements.

(2) If $x,y\varepsilon K_n$ with $x_{n-1} = y_n$, then $k_n(xy) = k_n x k_n y$.

(3) $k_n(x^{-1}) = (k_n x)^{-1}$

(4) If $v\varepsilon K_{n-1}$, then $k_n^v : K_v^n \longrightarrow K^n_{s_{n-2}d_{n-1}v}$ is an isomorphism and $k_n^{s_0^{n-1}a}$ is the identity for $a\varepsilon K_0$.

(5) If $p\varepsilon K_1$, then $k_n^{s_0^{n-2}p}\phi_n[s_0p_1,p] : K^n_{p_1} \longrightarrow K^n_{p_1}$ is the identity isomorphism, where $\phi_n[s_0p_1,p]$ is defined as in section 5.

(6) $k_nk_n = k_n$.

The proof follows from the definitions and 8.1.

Lemma 8.4.

If $x\varepsilon K_n$, then $I[x_n]k_nx(s_{n-2}x_{n-1}) = x$.

The proof follows from 8.1(a).

For convenience if $u\varepsilon K_{n-1}$ we write $h_n[u]$ for the isomorphism $h_n[u,s_0^{n-1}d_0^{n-1}u]$ as defined as in section 2.

Lemma 8.5.

If $u\varepsilon K_{n-1}$ and $x\varepsilon K^n_u$, then for $n\ge 2$ $(h_n[s_{n-2}d_{n-1}u]k_n^u x)^{d_0^{n-2}u} = h_n[u](x)$.

Proof. We use induction. Suppose the lemma is true for all $x\varepsilon K_v^{n-1}$. Hence the

lemma is true for all $x\varepsilon T^n_u = K^n_u \cap T_n$ ... (a).

Now let $u\varepsilon K_{n-1}$ $(n\geq 2)$, then we define ${}^0u = u$ and

$$ {}^iu = \begin{cases} u & \text{for } i\geq 1,\ n = 2 \\ s_{n-3}d_0{}^{(i-1)}u & \text{for } i\geq 1,\ n>2 \end{cases} $$

Hence ${}^ju = s_0^{n-2}d_0^{n-2}u$ for $j\geq n-2$.

Now let $x\varepsilon K^n_{{}^{(n-2)}u}$. For convenience we let ${}^{(n-2)}u = s_0^{n-2}p$ for $p\varepsilon K_1$. Then using 5.1 there exists a unique $y\varepsilon K^n_p$ such that $\phi[s_0p_1,p](y) = x$. Hence

$$
\begin{aligned}
(h[s_{n-2}d_{n-1}s_0^{n-2}p]k_nx)^p &= (k_nx)^p \quad \text{by 2.1(1).} \\
&= (k_n\phi[s_0p_1,p](y))^p \\
&= y^p \quad \text{by 8.2(5)} \\
&= h[s_0^{n-2}p]\phi[s_0p_1,p](y) \\
&= h[s_0^{n-2}p](x) \quad \text{as required.}
\end{aligned}
$$

We now suppose the lemma is true for $x\varepsilon K^n_{{}^ju}$ and $0\leq i<j\leq n-2$. Then to complete the proof we need to show that the lemma is true for $x\varepsilon K^n_{{}^iu}$.

Suppose $x\varepsilon K^n_{{}^iu}$, then it is clear that we can form a box $(-,w(x_1),\dots,w(x_{n-2}),x,s_{n-2}{}^iu,s_{n-2}{}^iu)$ where $w(x_l)\varepsilon T_n$ for $1\leq l\leq n-2$. Then we let $w(x)\varepsilon T_{n+1}$ fill our box. Thus $w(x)\varepsilon K^{n+1}_{s_{n-2}{}^iu}$ and $d_0w(x)\varepsilon K^n_{{}^{(i+1)}u}$. Now using (a) and our second inductive hypothesis we have that

$$(d_ih[s_{n-1}d_ns_{n-2}{}^iu]k_{n+1}w(x))^{d_0^{n-2}({}^iu)} = d_ih[s_{n-2}{}^iu]w(x) \text{ for } i = 0,\dots,n-2,n,n+1.$$

Hence by axiom T.2 of a T-complex we have that

$(h[s_{n-1}d_ns_{n-2}{}^iu]k_{n+1}w(x))^{d_0^{n-2}({}^iu)} = h[s_{n-2}{}^iu]w(x)$. Now using 2.1(4) and 7.3 we have that $(h[s_{n-2}d_{n-1}{}^iu]d_{n-1}k_{n+1}w(x))^{d_0^{n-2}({}^iu)} = h[{}^iu](x)$. But

$$
\begin{aligned}
d_{n-1}k_{n+1}w(x) &= (s_{n-2}{}^iu)x(s_{n-2}{}^iu)^{-1} \quad \text{by 8.2} \\
&= (s_{n-2}{}^iu)xI[{}^iu] \quad \text{by 8.1(a)} \\
&= k_n^{{}^iu}x \quad \text{as required.}
\end{aligned}
$$

Finally to start the induction for n = 2 let $x \varepsilon K_p^2$ $(p \varepsilon K_1)$. Then we need to show that $h[p](x) = (k_2 x)^p$. Using 5.1 there exists a unique element $y \varepsilon K^2_{s_0 p_1}$ such that $\phi[s_0 p_1, p](y) = x$. Then $(k_2 x)^p = (k_2 \phi[s_0 p_1, p](y))^p$

$$= y^p \quad \text{by } 8.2(5)$$

$$= h[p](x).$$

This completes the proof of lemma 8.5.

§9. The isomorphism theorem for T-complexes.

This section is concerned with finding sufficient conditions for deciding when two T-complexes are isomorphic.

Theorem 9.1.

Let (J,S) and (K,T) be T-complexes and suppose $f : N(J,S) \longrightarrow N(K,T)$ is a morphism of crossed complexes. Then f extends uniquely to a morphism $F : (J,S) \longrightarrow (K,T)$ of T-complexes.

Proof. We use induction. We suppose that the maps $F_i : J_i \longrightarrow K_i$ have been defined for $0 \le i \le n-1$ and satisfying :

(a) $F_i = f_i$ for $i = 0,1$.

(b) F_i maps thin elements to thin elements.

(c) If $x, y \varepsilon J_i$ with $x_{i-1} = y_i$, then $F_i(xy) = F_i x F_i y$.

(d) F_i commutes with the face maps.

(e) F_i is an extension of f_i.

Note that the properties (b) and (d) imply F_i commutes with the degeneracy maps and the elements I[x] for $x \varepsilon J_i$ where I[x] is defined as in section 1. Then to prove the theorem we need to extend f_n to a map $F_n : J_n \longrightarrow K_n$ which

satisfies the properties (a),...,(e).

We first uniquely extend $f_n^a : J_a^n(A) \longrightarrow K^n_{f_0 a}(A)$ to $f_n^a : J_a^n \longrightarrow K_a^n$ for $a\varepsilon J_0$ and $n\geq 3$. Using 3.3 it is sufficient to define a homomorphism $f_n^a : S_a^n \longrightarrow T_a^n$, where $S_a^n = J_a^n \cap S_n$ and $T_a^n = K_a^n \cap T_n$,such that $d_i f_n^a = f_{n-1}^a d_i$ for $i = 0,\ldots,n$. For $x\varepsilon S_a^n$ we define $f_n^a x$ to be the thin filler of the box $(-, f_{n-1}^a x_1, \ldots, f_{n-1}^a x_{n-2}, s_0^n f_0 a, s_0^n f_0 a)$. Then to show that f_n^a is well defined we need to show that $d_0 f_n^a x = f_{n-1}^a (d_0 x)$ for $a\varepsilon J_0$ and $x\varepsilon J_a^n$.However this follows from our inductive hypothesis and 4.1(6). Now using 7.10(4) we can uniquely extend f_n to a map $f_n^u : J_u^n \longrightarrow K^n_{f_{n-1}u}$ for all $u\varepsilon J_{n-1}$.

Now let $x\varepsilon J_n$, then we define $F_n x = I[F_{n-1}x_n](f_n^{s_{n-2}d_{n-1}x_n}(j_n x))s_{n-2}F_{n-1}x_{n-1}$ where $j_n : J_n \longrightarrow J_n$ is defined as in section 8.

With this definition it is clear that F_n satisfies the properties (b) and (c).. We now show that F_n satisfies property (d). Let $x\varepsilon J_n$, then

$$\begin{aligned}
d_n F_n x &= d_n I[F_{n-1}x_n] \quad \text{by 1.2(c)}\\
&= F_{n-1}x_n \quad \text{by 1.8}\\
d_{n-1}F_n x &= d_{n-1}s_{n-2}F_{n-1}x_{n-1} \quad \text{by 1.2(b)}\\
&= F_{n-1}x_{n-1}\\
d_{n-2}F_n x &= (F_{n-1}x_n)^{-1}F_{n-1}(x_n x_{n-2} x_{n-1}^{-1})F_{n-1}x_{n-1} \quad \text{by 8.2}\\
&= F_{n-1}(x_n^{-1} x_n x_{n-2} x_{n-1}^{-1} x_{n-1}) \quad \text{by our inductive hypothesis.}\\
&= F_{n-1}x_{n-2}
\end{aligned}$$

and for $i = 0,\ldots,n-3$

$$\begin{aligned}
d_i F_n x &= I[F_{n-2}x_{n,i}]F_{n-1}j_{n-1}x_i s_{n-3}F_{n-2}x_{n-1,i}\\
&= F_{n-1}(I[x_{n,i}]j_{n-1}x_i s_{n-3}x_{n-1,i}) \quad \text{by our inductive hypothesis.}\\
&= F_{n-1}x_i \quad \text{by 8.4.}
\end{aligned}$$

It remains to show that F_n satisfies property (e). ~~We need to~~ It is sufficient to show that for $u\varepsilon J_{n-1}$ and $x\varepsilon J_u^n$, $F_n x = f_n^u(x)$. We let $v = s_{n-2}d_{n-1}u$, $d_0^{n-1}v = a\varepsilon J_0$ and

$d_0^{n-2}u = p \varepsilon J_1$. Now consider the following diagram.

$$\begin{tikzcd}
 & & & y \arrow[r, mapsto] & y^p \\
J^n_u \arrow[r, leftrightarrow, "j_n"] \arrow[d, dashed, "F_n"'] & J^n_v \arrow[r, leftrightarrow, "h[v]"] \arrow[d, "f^v_n"'] \arrow[dr, phantom, "="] & J^n_a \arrow[r, leftrightarrow] \arrow[d, "f^a_n"'] & J^n_{d_0p} \arrow[d, "f^{d_0p}_n"] \\
K^n_{F_{n-1}u} \arrow[r, leftrightarrow, "k_n"] & K^n_{F_{n-1}v} \arrow[r, leftrightarrow, "{h[F_{n-1}v]}"] & K^n_{f_0a} \arrow[r, leftrightarrow] & K^n_{f_0d_0p} \\
 & & & w \arrow[r, mapsto] & w^{f_1p}
\end{tikzcd}$$

Then it is clear that the end square of our diagram is commutative and so by 8.5 we have that

$$F_nx = \left(h[F_{n-1}u]\right)^{-1} f_n^{d_0p} h[u](x)$$
$$= f_n^u(x) \quad \text{by } 7.8(4).$$

For uniqueness see appendix P.76.

This completes the proof of theorem 9.1.

Using theorem 9.1 we deduce the following theorem.

Theorem 9.2.

Let $f : (J,S) \longrightarrow (K,T)$ be a morphism of T-complexes. Then the following two statements are equivalent.

(1) f is an isomorphism of T-complexes.

(2) $f : N(J,S) \longrightarrow N(K,T)$ is an isomorphism of crossed complexes.

§10. The T-complex addition lemma.

In this section we define the T-complex addition lemma which is analogous to the homotopy addition lemma (see Hu [6]).

Let (K,T) be a T-complex. Recall that for convenience we write $h_n[u]$ for the isomorphism $h_n[u, s_0^{n-1}d_0^{n-1}u]$ whenever $u \varepsilon K_{n-1}$.

For $n \geq 3$ we define a function $\mu_n : K_n \longrightarrow K_n$ where p_n is

$$x \longmapsto p_n h_n[s_{n-2}d_{n-1}x_n]k_n x$$

defined as in 4.1(7) and for n = 2 we let $\mu_2 = k_2$. Then by 2.1 and 8.3 we have the following lemma.

Lemma 10.1.

(1) μ_n maps thin elements to thin elements.

(2) If $x, y \in K_n$ with $x_{n-1} = y_n$, then $\mu_n(xy) = \mu_n x \mu_n y$.

(3) If $x \in K_n$ $(n \geq 3)$, then $\mu_n x \in K^n_a(A)$ where $a = d_0^{n-1} s_{n-2} d_{n-1} x_n \in K_0$.

Now consider the following diagram.

$$\begin{tikzcd}
K_n \arrow[r, "\mu_n"] \arrow[d, "d_i"'] & (K^n_a(A))_{a\in K_0} \arrow[d, "\delta_n"] \\
K_{n-1} \arrow[r, "\mu_{n-1}"'] & (K^{n-1}_a(A))_{a\in K_0}
\end{tikzcd}$$

Then we may ask: if $x \in K_n$, are there any relationships between the elements of the set $\{\delta_n\mu_n x, \mu_{n-1}x_0, \ldots, \mu_{n-1}x_n\}$?

Lemma 10.2 The T-complex addition lemma.

Let (K,T) be a T-complex. Then if $x \in K_n$ $(n \geq 2)$ we have that :

$\delta_2\mu_2 x = x_2 x_0 x_1^{-1}$

$\delta_3\mu_3 x = \mu_2 x_0 (\mu_2 x_2 \mu_2 x_1^{-1} \mu_2 x_3^{-1})^{d_2 x_3}$

$\delta_n\mu_n x = (\prod_{i=0}^{n-3} \mu_{n-1} x_i^{(-1)^i})(\mu_{n-1} x_{n-2}^{(-1)^{n-2}} \mu_{n-1} x_{n-1}^{(-1)^{n-1}} \mu_{n-1} x_n^{(-1)^n})^p$ for $n>3$,

where $p = d_0^{n-3} d_{n-1} x_n \in K_1$.

Proof. Let $x \in K_n$, then for $n \geq 3$ we have that:

$\delta_n\mu_n x = \delta_n p_n h_n k_n x = p_{n-1}\delta_n h_n k_n x$ by 4.1(7)

$= (\prod_{i=0}^{n-3} \mu_{n-1} x_i^{(-1)^i})(p_{n-1}h_{n-1}[d_{n-1}x_n](x_n x_{n-2} x_{n-1}^{-1})^{(-1)^{n-2}}$ by 2.1(4) and 8.2

Now using 8.5 we see that

$$p_{n-1}h_{n-1}[d_{n-1}x_n](x_nx_{n-2}x_{n-1}^{-1}) = (p_{n-1}h_{n-1}[s_{n-3}d_{n-2}d_{n-1}x_n]k_{n-1}(x_nx_{n-2}x_{n-1}^{-1}))^p$$
$$= (\mu_{n-1}x_n\mu_{n-1}x_{n-2}\mu_{n-1}x_{n-1}^{-1})^p \quad \text{by 2.1 and 8.3(2).}$$

Hence we have the required result for $n\geq 3$.

Finally if $x\varepsilon K_2$, then $d_0k_2x = d_0(s_0x_2xI[x_1])$
$$= x_2x_0x_1^{-1}$$

This completes the proof of lemma 10.2.

We now wish to state the T-complex addition lemma in a slightly different form. Let $x\varepsilon K_n$. Then we define $O(x) = \prod_{i=0}^{n-1} d_2\ldots d_nx^i$ (recall that $x^0 = x$ and $x^i = s_{n-2}d_0x^{i-1}$) We now define a new function for $n\geq 2$

$$\mu_n^0 : K_n \longrightarrow K_n \qquad \text{for } n\geq 3.$$
$$x \longrightarrow (\mu_nx)^{-O(s_{n-2}d_{n-1}x_n)}$$

Lemma 10.3.

If $x\varepsilon K_n$, then for $n\geq 3$ we have that :

$$\delta_3\mu_3^0x = \mu_2^0x_0^{-(d_2x_3)}\mu_2^0x_2\mu_2^0x_1^{-1}\mu_2x_3^{-1}$$
$$\delta_n\mu_n^0x = \mu_{n-1}^0x_0^{-(d_2\ldots d_nx)}\left(\prod_{i=1}^{n}\mu_{n-1}^0x_i^{(-1)^i}\right) \quad \text{for } n>3.$$

The proof is straightforward using the fact that the image of δ_2 acts trivially on $(K_a^n(A))_{a\varepsilon K_0}$ for $n\geq 3$ (see section 7) and lemma 10.2.

We remark that for $x\varepsilon T_n$, $\mu_nx = s_o^na$ the identity of $K_a^n(A)$ where $a = d_0^{n-1}s_{n-2}d_{n-1}x_n$.

§11. A functor from crossed complexes to T-complexes.

In this section we use lemma 10.3 to define a functor from the

category of crossed complexes to the category of T-complexes.

Let $c : \ldots \longrightarrow c_n \xrightarrow{\delta_n} c_{n-1} \longrightarrow \cdots \longrightarrow c_2 \xrightarrow{\delta_2} c_1 \underset{\delta_1}{\overset{\delta_0}{\rightrightarrows}} c_0$

be a crossed complex . Then for $a \varepsilon c_0$ we write 1_a for the identity of the group $c_n(a)$, $n \geq 1$.

We will now construct , from a crossed complex c, a simplicial set $Dc = ((D_n c)_{n\geq 0}, d_i, s_i)$ as follows. We remark that our Dc = Blakers Kc (cf p. 49).

Lemma 11.1.

If c is a crossed complex, then there is a unique graded set $(D_n c)_{n\geq 0}$ and a unique graded set of maps $(d_0, \ldots, d_n : D_n c \longrightarrow D_{n-1} c)$ such that :

(a) $D_0 c = c_0$

(b) $D_1 c = c_1$

(c) $d_i = \delta_i : D_1 c \longrightarrow D_0 c \quad i = 0,1$

(d) For $n \geq 2$ elements of $D_n c$ are of the form $\underline{c}(x) = (\underline{c}_0(x_0), \ldots, \underline{c}_n(x_n); x)$ and satisfying :

(1) $\underline{c}_i(x_i) \varepsilon D_{n-1} c$ for $0 \leq i \leq n$ and $d_i \underline{c}_j(x_j) = d_{j-1} \underline{c}_i(x_i)$, $j > i$.

(2) $x \varepsilon c_n(d_1 \ldots d_{n-1} \underline{c}_n(x_n))$

(3)
$$\delta_n x = \begin{cases} x_2 x_0 x_1^{-1} & \text{for } n = 2 \\ x_0^{p^{-1}} x_2 x_1^{-1} x_3^{-1} & \text{for } n = 3 \text{ where } p = d_2 c_3(x_3) \\ x_0^{p^{-1}} \left(\prod_{i=1}^{n} x_i^{(-1)^i}\right) & \text{for } n > 3 \text{ where } p = d_2 \ldots d_{n-1} \underline{c}_n(x_n) \end{cases}$$

(4) $d_i \underline{c}(x) = \underline{c}_i(x_i)$ for $0 \leq i \leq n$.

The proof is straightforward using induction.

Lemma 11.2.

There is a unique graded set of maps $(s_0, \ldots, s_{n-1} : D_{n-1} c \longrightarrow D_n c)_{n\geq 1}$ such that:

(a) For $n = 1$ and $a \varepsilon c_0$, $s_0 a = 1_a$

(b) For $n > 1$ and $\underline{c}(x) \varepsilon D_{n-1} c$ we have that

$s_i \underline{c}(x) = (s_{i-1} d_0 \underline{c}(x), \ldots, s_{i-1} d_{i-1} \underline{c}(x), \underline{c}(x), \underline{c}(x), s_i d_{i+1} \underline{c}(x), \ldots, s_i d_{n-1} \underline{c}(x); 1_b)$

where $b = d_1 \ldots d_{n-1} \underline{c}(x)$.

The proof is straightforward using induction.

Corollary 11.3.

If c is a crossed complex , then Dc is a simplicial set.

Let c be a crossed complex. Then we define $Tc = (T_n c)_{n\geq 1}$ to be the graded subset of Dc such that :

$$T_n c = \left\{ \begin{array}{l} 1_a \text{ for } a\varepsilon c_0 \text{ and } n = 1 \\ \\ \underline{c}(x)\varepsilon D_n c \ : \ x = 1_a \text{ for some } a\varepsilon c_0 \text{ and } n\geq 2. \end{array} \right\}$$

We will call elements of Tc thin.

Theorem 11.4.

If c is a crossed complex , then (Dc,Tc) is a T-complex.

Proof. We check that (Dc,Tc) satisfies the axioms for a T-complex. Axioms T.1 and T.3 follow immediately from the definition of (Dc,Tc). Hence to prove the theorem we need to verify axiom T.2 , that is that every box in Dc has a unique thin filler. We begin by considering the case n≥5.

Let $(\underline{c}_0(x_0),\ldots,\underline{c}_{i-1}(x_{i-1}),-,\underline{c}_{i+1}(x_{i+1}),\ldots,\underline{c}_n(x_n))$ constitute a box in $D_{n-1}c$, that is $\underline{c}_j(x_j)\varepsilon D_{n-1}c$ for $0\leq j\neq i\leq n$. Suppose our box has a thin filler $\underline{c}(1_b)$, where $b = d_1\ldots d_{n-1}\underline{c}_n(x_n)$, with $d_i\underline{c}(1_b) = \underline{c}(y)\varepsilon D_{n-1}c$. We first show that $\underline{c}(1_b)$ is the only thin filler.

Firstly we see that

$$y = \left\{ \begin{array}{ll} \left(\left(\prod_{j=1}^{n} x_j^{(-1)^j}\right)^{-1}\right)^p & \text{where } p = d_2\ldots d_{n-1}\underline{c}_n(x_n) \text{ and } i = 0 \, . \\ \\ \left(x_0^{p^{-1}}\left(\prod_{\substack{j=1\\ j\neq i}}^{n} x_j^{(-1)^j}\right)\right)^{(-1)^{i+1}} & \text{for } 1\leq i\leq n, \text{ where } p = d_2\ldots d_{n-1}\underline{c}_n(x_n) \end{array} \right\} \ldots(a)$$

and secondly $\underline{c}(y)$ has as its shell

$(d_{i-1}\underline{c}_0(x_0),\ldots,d_{i-1}\underline{c}_{i-1}(x_{i-1}),d_i\underline{c}_{i+1}(x_{i+1}),\ldots,d_i\underline{c}_n(x_n))$. Hence $\underline{c}(y)$ is uniquely defined by the set $\{\underline{c}_j(x_j)\}_{0\le j\ne i\le n}$ and so if our box has a thin filler it is unique.

To show our box has a thin filler it is sufficient to show that the element $\underline{c}(y) = (d_{i-1}\underline{c}_0(x_0),\ldots,d_{i-1}\underline{c}_{i-1}(x_{i-1}),d_i\underline{c}_{i+1}(x_{i+1}),\ldots,d_i\underline{c}_n(x_n);y)$ is an element of $D_{n-1}c$ where y is defined as in (a). We now prove that $\underline{c}(y)$ is a well defined element of $D_{n-1}c$.

For $0\le j\ne i\le n$ we let $\underline{c}_j(x_j) = (\underline{c}_0(x_{j,0}),\ldots,\underline{c}_{n-1}(x_{j,n-1});x_j)$. Then

$$\delta_{n-1}x_j = x_{j,0}^{p(j)^{-1}}\left(\prod_{l=1}^{n-1} x_{j,l}^{(-1)^l}\right) \quad \ldots\text{(b)} \quad \text{where } p(j) = d_2\ldots d_{n-1}\underline{c}_j(x_j).$$

We note that :

$$\left.\begin{array}{l} x_{j,1} = x_{1,j-1} \text{ for } j>1 \text{ and } i\ne j,1 \text{ and} \\ p(2) = p(3) = \ldots = p(n) = p = d_2\ldots d_{n-1}c_n(x_n). \end{array}\right\} \ldots\text{(c)}$$

Now for $\underline{c}(y)$ to be a well defined element of $D_{n-1}c$ we need to show that

$$\delta_{n-1}y = \begin{cases} x_{1,0}^{p(0)^{-1}}\left(\prod_{j=1}^{n-1} x_{j+1,}^{(-1)^i}\right) \text{ for } i = 0 \\ \\ x_{0,i-1}^{p(i)^{-1}}\left(\prod_{j=1}^{i-1} x_{j,i-1}^{(-1)^j}\right)\left(\prod_{j=i+1}^{n} x_{j,i}^{(-1)^j}\right) \quad \text{for } 1\le i\le n. \end{cases}$$

However since $n\ge 5$ we are working in abelian groups (see axiom C.2 of a crossed complex) $c_{n-2}(d_1p)$ and so by (a),(b) and (c) it is sufficient to show that $x_{0,0}^{p(0)^{-1}p(2)^{-1}} = x_{0,0}^{p(1)^{-1}}$.

Let $d_3\ldots d_{n-1}c_j(x_j) = (q_0,q_1,q_2;z)$ for any $2\le j\ne i\le n$. Then $p(k) = q_k$ $0\le k\le 2$ Furthermore $\delta_2 z = q_2q_0q_1^{-1}$ and so we have that $x_{0,0}^{p(1)^{-1}p(2)^{-1}\delta_2 z} = x_{0,0}^{p(1)^{-1}}$ whenever $x_{0,0}\varepsilon c_{n-2}(d_0p_1)$. But the image of δ_2 acts trivially on c_n for $n\ge 3$ (see axiom C.5) and so we have the required result. Thus for $n\ge 5$ we have shown that every n-1 dimensional box has a unique thin filler.

The low dimensional cases ($n\le 4$) are proved by a similiar method to the

one used for the cases $n \geq 5$ and the following lemma.

<u>Lemma 11.5.</u>

If c is a crossed complex, then for $a \varepsilon c_0$ $\delta_3 c_3(a)$ is in the centre of $c_2(a)$.

<u>Proof.</u> Let $x \varepsilon c_3(a)$. Then by axiom C.6 of a crossed complex $\delta_2 \delta_3 x = 1_a$. Hence, by axiom C.5, for $y \varepsilon c_2(a)$ we have that

$y = y^{(1_a)} = y^{\delta_2 \delta_3 x} = (\delta_3 x)^{-1} y \delta_3 x$ as required.

For the low dimensional cases we restrict our attention to the following case. Let $(-, \underline{c}_1(x_1), \ldots, \underline{c}_4(x_4))$ constitute a box where $\underline{c}_i(x_i) \varepsilon D_3 c$ for $1 \leq i \leq 4$. Then we let $\underline{c}(y) = (d_0 \underline{c}_1(x_1) d_0 \underline{c}_2(x_2), d_0 \underline{c}_3(x_3), d_0 \underline{c}_4(x_4); y)$ where $y = (x_4^{-1} x_3 x_2^{-1} x_1)^p$ and $p = d_2 d_3 \underline{c}_4(x_4)$. Then we are required to show that $\underline{c}(y) \varepsilon D_3 c$.

Let $p(i) = d_2 d_3 \underline{c}_i(x_i)$ for $0 \leq i \leq 4$, then $p(2) = p(3) = p(4) = p$. Thus it is sufficient to show that

$\delta_3 (x_4^{-1} x_3 x_2^{-1} x_1)^p = x_{1,0}^{p(0)^{-1}} x_{3,0} (x_{2,0})^{-1} (x_{4,0})^{-1}$. But we have that

$\delta_3 (x_i)^p = (x_{i,0}^{p(i)^{-1}} x_{i,2} (x_{i,1})^{-1} (x_{i,3})^{-1}$ for $1 \leq i \leq 4$. (Recall that we write $x_{i,j}$ for $d_j d_i x$.) Then

$\delta_3 (x_4^{-1} x_3 x_2^{-1} x_1) = x_{4,3} \delta_3 x_1 x_{4,1} (x_{4,2})^{-1} (\delta_3 x_2)^{-1} (x_{4,0}^{p^{-1}})^{-1} \delta_3 x_3$ by 11.5

$$
\begin{aligned}
&= x_{4,3} x_{1,0}^{p(1)^{-1}} x_{1,2} (x_{2,2})^{-1} (x_{2,0}^{p^{-1}})^{-1} (x_{4,0}^{p^{-1}})^{-1} \delta_3 x_3 \\
&= (x_{1,0})^{p(0)^{-1} p(2)^{-1} \delta_2 x_{4,3}} x_{1,2} (x_{2,2})^{-1} (x_{2,0}^{p^{-1}})^{-1} (x_{4,0}^{p^{-1}})^{-1} \delta_3 x_3 \quad \ldots * \\
&= (x_{1,0})^{p(0)^{-1} p(2)^{-1} \delta_3 x_3 \delta_2 x_{4,3}} x_{1,2} (x_{2,2})^{-1} (x_{2,0}^{p^{-1}})^{-1} (x_{4,0}^{p^{-1}})^{-1} \\
&= (x_{1,0})^{p(0)^{-1} p^{-1}} x_{3,0}{}^{p^{-1}} (x_{2,0}^{p^{-1}})^{-1} (x_{4,0}^{p^{-1}})^{-1} \quad \text{as required.}
\end{aligned}
$$

It remains for us to justify *. Since $\delta_2 x_{4,3} = p(2) p(0) p(1)^{-1}$ we have that

$x_{1,0}^{p(1)^{-1}} = (x_{1,0})^{p(0)^{-1} p(2)^{-1} \delta_2 x_{4,3}} = (x_{4,3})^{-1} (x_{1,0})^{p(0)^{-1} p(2)^{-1}} (x_{4,3})$ by C.5.

This completes the proof of theorem 11.4.

Lemma 11.6.

If $f : c \longrightarrow \bar{c}$ is a morphism of crossed complexes, then for $n \geq 0$ there is a unique family of maps $D_n f : D_n c \longrightarrow D_n \bar{c}$ and satisfying :

(a) $D_i f = f_i$ for $i = 0,1$.

(b) For $n \geq 2$ and $\underline{c}(x) = (\underline{c}_0(x_0),\dots,\underline{c}_n(x_n);x) \varepsilon D_n c$ we have that $D_n f\underline{c}(x) = (D_{n-1}f\underline{c}_0(x_0),\dots,D_{n-1}f\underline{c}_n(x_n);f_n x)$ is a well defined element of $D_n\bar{c}$.

The proof is straightforward using induction.

Corollary 11.7.

If $f : c \longrightarrow \bar{c}$ is a morphism of crossed complexes , then $Df : (Dc,Tc) \longrightarrow (D\bar{c},T\bar{c})$ is a morphism of T-complexes.

Thus we have a well defined functor $D : C \longrightarrow T$.

§12. The equivalence of categories.

In this final section of chapter 1 we show that the categories of T-complexes and crossed complexes are equivalent.

By our previous work it is sufficient to show that for c an object of C, then c is isomorphic to N(Dc,Tc) and for (K,T) an object of T, then (K,T) is isomorphic to (DN(K,T),TN(K,T)). To enable us to do this we need the following technical results.

Lemma 12.1.

Let c be a crossed complex and suppose $\underline{c}(x)$, $\underline{c}(y) \varepsilon D_n c$ with $d_{n-1}\underline{c}(x) = d_n\underline{c}(y)$,

then we have that :

(a) $\underline{c}(x)\underline{c}(y) = \underline{c}(xy)$

(b) If $d_{n-1}\underline{c}(x) = d_n\underline{c}(x) = \underline{c}(u)\varepsilon D_{n-1}c$, then $d_0F[\underline{c}(x),\underline{c}(u)] = \underline{c}((x^{(-1)^{n-1}})^p$

where $p = d_2\ldots d_{n-1}\underline{c}(u)$ and $F[\underline{c}(x),\underline{c}(u)]$ is defined as in section 2.

The proof follows immediately from the definitions.

Corollary 12.2.

Let c be a crossed complex and suppose $\underline{c}(x)\varepsilon D_nc$ with $d_{n-1}\underline{c}(x) = d_n\underline{c}(x) = \underline{c}(u)$, then $h[\underline{c}(u),s_0^{n-1}d_0^{n-1}\underline{c}(u)]\underline{c}(x) = \underline{c}(x^P)$ where $P = p_0\ldots p_{n-2}$ and for $0\le i\le n-2$ $p_i = d_2\ldots d_{n-1}\underline{c}(u)^i$ (Recall that $\underline{c}(u)^i = s_{n-2}d_0\underline{c}(u)^{i-1}$ and $\underline{c}(u)^0 = \underline{c}(u)$).

Lemma 12.3.

Let c be a crossed complex and suppose $p\varepsilon c_1$ and $\underline{c}(x) = (\underline{c}_0(x_0),1_{p_1},\ldots,1_{p_1};x)$ is an element of D_nc ($n\ge 2$), then $\underline{c}(x)^p = (\underline{c}_0(x_0)^p,1_{p_0},\ldots,1_{p_0};x^p)$.

Proof. By 7.3 we have that $\underline{c}(x)^p = (\underline{c}_0(x_0)^p,1_{p_0},\ldots,1_{p_0};w)$ where $w\varepsilon c_n(p_0)$. Hence we need to show that $w = x^p$.

$$\begin{aligned}\text{But } \underline{c}(x)^p &= h[s_0^{n-2}p,s_0^{n-1}p_0]\phi[s_0p_1,p]\underline{c}(x) \quad \text{(see section 7)}\\ &= h[s_0^{n-2}p,s_0^{n-1}p_0]\underline{c}^1(x) \quad \text{by 12.1 and our definition of } \phi\\ &= \underline{c}^2(x)^P (\text{where } P = p_0\ldots p_{n-2} \text{ and } p_i = d_2\ldots d_{n-1}(s_0^{n-2}p)^i) \text{ by 12.2.}\end{aligned}$$

But $p_0 = p_1 = \ldots = p_{n-3} = 1_{p_1}$ and $p_{n-2} = p$ as required.

Lemma 12.4.

Let c be a crossed complex. Then N(Dc,Tc) is isomorphic to c.

Proof. $N_i(Dc,Tc) = c_i$ for $i = 0,1$ and for $n\ge 2$ we have that $N_n(Dc,Tc) = \{\underline{c}(x)\varepsilon D_nc : c(x) = (\underline{c}_0(x_0),1_a,\ldots,1_a;x),a\varepsilon c_0\}$. Then for $(\underline{c}_0(x_0),1_a,\ldots,1_a;x)\varepsilon N_n(Dc,Tc)$ we have that $\delta_nx = x_0$ and so $\delta_{n-1}x_0 = 1_a$ by axiom C.6 of a crossed complex. Hence x may take all values in c_n and so we can define a bijection $i_n(c) : N_n(Dc,Tc) \longrightarrow c_n$. Then by 12.1(a) and 12.3

$$\underline{c}(x) \longmapsto x$$

it is clear that $(i_n(c))_{n\ge 0}$ is an isomorphism of crossed complexes.

This completes the proof of lemma 12.4.

Lemma 12.5.

Let (K,T) be a T-complex. Then (K,T) is isomorphic to (DN(K,T),TN(K,T)).

Proof. By 12.4 N(K,T) is isomorphic to N(DN(K,T),TN(K,T)). Then by 9.1 and 9.2 we have the required result.

This completes the proof that the category of T-complexes is equivalent to the category of crossed complexes.

We note that Dakin [1] has proved that the category of T-complexes of rank 2 is equivalent to the category of crossed modules over a groupoid.

CHAPTER 2

Special filtered Kan complexes.

§1. Introduction.

In this chapter we define the notion of a special filtered Kan complex. Given a filtered Kan complex $\underline{K} : K^0 \subset K^1 \subset \dots \subset K^n \subset \dots$ we show how to associate a Kan complex $R(\underline{K})$ and a T-complex $\rho(\underline{K})$ to such an object.

We relate this theory to work of R. Brown and P.J. Higgins [3] . They construct for a filtered space $\underline{X} : X_0 \subset X_1 \subset \dots \subset X_n \subset \dots$ a cubical Kan complex $R(\underline{X})$ and by imposing a relation of filtered homotopy on $R(\underline{X})$ obtain a (cubical) T-complex, provided each loop in X_0 is contractible in X_1. We show in the simplicial context that the analogous $R(\underline{X})$ can be given the structure of a special filtered Kan complex $\underline{K}$ such that $R(\underline{K}) = R(\underline{X})$, $\rho(\underline{K}) = \rho(\underline{X})$.

For Brown-Higgins the main technical tool is that the projection $p : R(\underline{X}) \longrightarrow \rho(\underline{X})$ is a Kan fibration. Similarly, we find the construction of $R(\underline{K})$ and $\rho(\underline{K})$ as simplicial sets presents no problems, but the hardest work is in proving $p : R(\underline{K}) \longrightarrow \rho(\underline{K})$ is a Kan fibration. This result is used in proving that the natural definition of thin elements in $\rho(\underline{K})$ does make $\rho(\underline{K})$ a T-complex.This gives the important geometric example of a T-complex.

A consequence of our results is that, in analogy to the Brown-Higgins result, the crossed complex associated to the simplicial T-complex $\rho(\underline{X})$ is the homotopy crossed complex $\pi_*(\underline{X})$.

Finally we note that Blakers [7] defines simplicial sets $S(X)$ for a

filtered space and $K(c)$ for a crossed complex c, and considers a projection $S(X) \longrightarrow K(\pi_*(X))$. In fact $S(X) = R(\underline{X})$ and our results show $K(\pi_*(\underline{X})) = \rho(\underline{X})$.

Throughout this chapter we use the following notations. If K is a simplicial set and if $x \in K_n$, then we write (x) for the subcomplex of K generated by x. Thus $\Delta(n) = (\delta^n)$. For convenience we write (i) for $(d_i\delta^1)$ $i = 0,1$, I for $\Delta(1)$ and I^r for $I\text{x}\ldots\text{x}I$, $\dot{I}$ for $\Delta(1,0)$, $\dot{\Delta}(n)$ for $\Delta(n,n-1)$. If $f : \wedge^k(n)\text{x}I \longrightarrow L$ is a simplicial map then we write $d_i f$ for $\Delta(n-1)\text{x}I \xrightarrow{\tilde{d}_i\text{x}1} \wedge^k(n)\text{x}I \longrightarrow L$ where $\tilde{d}_i : \Delta(n-1) \longrightarrow \wedge^k(n)$ is the standard inclusion. A similiar convention applies for $f : \Delta(n)\text{x}I \longrightarrow L$.

The nondegenerate elements $(s_i\delta^n, s_n\ldots s_{i+1}s_{i-1}\ldots s_0\delta^1)$ $i = 0,\ldots,n$ in dimension $n+1$ of $\Delta(n)\text{x}I$ will be written as a_i $i = 0,\ldots,n$ respectively. Thus if $f : \Delta(n)\text{x}I \longrightarrow L$ is a simplicial map , then f is determined by its values on the a_i $i = 0,\ldots,n$.

§2. Definitions and examples.

We begin by introducing the notion of collapsing in a simplicial set. Let L be a simplicial set. Then a non degenerate element $x \in L_n$ is said to have a free face $d_i x$ if $d_i x$ is non degenerate and is a face of no other non degenerate element of L. Then $L \backslash ((x) \backslash \cup_{k \neq i} (d_k x)) = L_0$ is a simplicial set with $L_0 \subset L$. The process of passing from L to L_0 is called an elementary collapse written $L \searrow^e L_0$. If there is a sequence of elementary collapses $L \searrow^e L_0 \searrow^e \ldots \searrow^e L_p$ then we say that L collapses to L_p and we write $L \searrow L_p$.

Lemma 2.1.

$\Delta(n)\text{x}I \searrow \Delta(n)\text{x}(1) \cup \dot{\Delta}(n)\text{x}I$

Proof. Let $a_0,\ldots,a_n$ be the non degenerate elements in dimension n+1 of $\Delta(n)\times I$. Then d_0a_0 is a free face of a_0 and so $\Delta(n)\times I \searrow^{e} (\Delta(n)\times I)\setminus((a_0)\setminus \bigcup_{i\neq 0}(d_i a_0)) = L_0$. Then in L_0, a_1 has a free face d_1a_1 and so $L_0 \searrow^{e} L_0\setminus((a_1)\setminus\bigcup_{i\neq 1}(d_i a_1)) = L_1$. Proceeding in this way we obtain the required result.

Lemma 2.2.

For $n\geq 0$ and $i = 0,1$ let

$A^i(n) = \Delta(n)\times(i)\times I \cup \Delta(n)\times I\times\dot{I} \cup \dot{\Delta}(n)\times I^2$

$B^i(n) = \Delta(n)\times I\times(i) \cup \Delta(n)\times\dot{I}\times I \cup \dot{\Delta}(n)\times I^2$

and for $n\geq 0$ and $k = 0,\ldots,n$ let

$C^k(n) = \Delta(n)\times\dot{I}^2 \cup \Lambda^k(n)\times I^2$ where $\dot{I}^2 = I\times\dot{I} \cup \dot{I}\times I$

Then $\Delta(n)\times I^2$ collapses to each of the following :

$A^0(n), A^1(n), B^0(n), B^1(n), C^0(n),\ldots,C^n(n)$.

Proof. We prove that $\Delta(n)\times I^2 \searrow B^1(n)$, the other cases being similiar.

$\Delta(n)\times I^2$ has only degenerate elements above dimension n+2. The non degenerate elements of dimension n+2 are the non degenerate elements of $(a_i)\times I$ $i = 0,\ldots,n$. Using lemma 2.1 we may collapse $(a_i)\times I \searrow ((a_i)\times(1) \cup (\dot{a}_i)\times I)$ for $i = 0,\ldots,n$. It now remains for us to collapse the internal faces $d_{i+1}a_i$ of a_i. But again by lemma 2.1 $(d_{i+}\, a_i)\times I \searrow ((d_{i+1}a_i)\times(1) \cup \dot{(d_{i+1}a_i)}\times I) \subset B^1(n)$ $i = 0,\ldots,n$. This completes the proof.

We now introduce the notions of a filtered Kan complex and a special filtered Kan complex.

Definition 2.3.

By a filtered Kan complex $\underline{K}$ is meant a sequence of Kan complexes K^i $i = 0,1,\ldots$ such that $K^i \subset K^{i+1}$ for $i\geq 0$.

Definition 2.4.

By a special filtered Kan complex is meant a filtered Kan complex $\underline{K}$ and satisfying :

(a) $d_i K_n^j \subset K_{n-1}^{j-1}$ for $j \geq n \geq 1$ and $0 \leq i \leq n$.

(b) Every simplicial map $f : \dot{\Delta}(2) \longrightarrow K^0$ extends to a simplicial map $\hat{f} : \Delta(2) \longrightarrow K^1$.

We note that by condition (b) of 2.4, a simplicial map $f : \dot{I}^2 \longrightarrow K^0$ extends to a simplicial map $\hat{f} : I^2 \longrightarrow K^1$.

Let $\underline{K}$ be a special filtered Kan complex. Then the graded set $R(\underline{K})$ such that $R(\underline{K})_n = K_n^n$ inherits a simplicial structure from $\underline{K}$, and it is clear that $R(\underline{K})$ is a Kan complex.

As a key example of a special filtered Kan complex consider the following.

Example 2.5.

Let R^{n+1} denote the n+1 dimensional euclidean space with coordinate system $x_0,\ldots,x_n$. Recall that the n-simplex Δ^n is the subset of R^{n+1} consisting of the points $(x_0,\ldots,x_n)$ of R^{n+1} such that $x_0 + \ldots + x_n = 1$, $x_i \geq 0$ $(0 \leq i \leq n)$.

We let $\underline{\Delta}^n : \Delta^{n,0} \subset \Delta^{n,1} \subset \ldots \subset \Delta^{n,n} = \Delta^n$ be the standard filtration of Δ^n, that is $\Delta^{n,r}$ is the union of all the faces of Δ^n whose dimension are $\leq r$. For $i = 0,\ldots,n$ let $\tilde{d}_i : \Delta^n \longrightarrow \Delta^{n+1}$ and $\tilde{s}_i : \Delta^n \longrightarrow \Delta^{n-1}$ be maps defined by

$\tilde{d}_i(x_0,\ldots,x_n) = (x_0,\ldots,x_{i-1},0,x_i,\ldots,x_n)$ and

$\tilde{s}_i(x_0,\ldots,x_n) = (x_0,\ldots,x_{i-1},x_i+x_{i+1},x_{i+2},\ldots,x_n)$..

Now let $\underline{X} : X_0 \subset X_1 \subset \ldots$ be a filtered space satisfying the condition that loops in X_0 are contractible in X_1 ...(*). We construct a special filtered

Kan complex from $\underline{X}$ as follows.

For m≥0 we let $R^m(\underline{X})$ be the Kan complex which in dimension n consists of filtered maps $f : \underline{\Delta}^n \longrightarrow \underline{X}$ such that $f(\Delta^n) \subset X_m$. Then for $f \varepsilon R^m_n(\underline{X})$ we define $d_i f = f\tilde{d}_i$ and $s_i f = f\tilde{s}_i$. Thus it is clear that $R^m(\underline{X}) \subset R^{m+1}(\underline{X})$ for m≥0, and $d_i R^j_n(\underline{X}) \subset R^{j-1}_{n-1}(\underline{X})$ for j≥n≥1 and i = 0,...,n. Further by condition (*) we see that any simplicial map $f : \dot{\Delta}(2) \longrightarrow R^0(\underline{X})$ extends to a simplicial map $f : \Delta(2) \longrightarrow R^1(\underline{X})$. Hence we have constructed, from a filtered space $\underline{X}$, a special filtered Kan complex $K(\underline{X}) : R^0(\underline{X}) \subset R^1(\underline{X}) \subset \dots \subset R^m(\underline{X}) \subset \dots$. We let $R(K(\underline{X})) = ((R(K(\underline{X}))_n)_{n\geq 0}, d_i, s_i)$ be the Kan complex where $R(K(\underline{X}))_n = R^n_n(\underline{X})$.

We now define an equivalence relation on $R(K(\underline{X}))$ by saying two elements x,y of $R(K(\underline{X}))_n$ are filtered homotopic if they are homotopic through filtered maps : $\underline{\Delta}^n \longrightarrow \underline{X}$, that is $\Delta^{n,r} x I \longrightarrow X_r$. These sets of homotopy classes form a simplicial set which we denote by $\rho(\underline{X})$.

We now apply the notion of filtered homotopy to a special filtered Kan complex.

<u>Definition 2.6.</u>

Let $\underline{K}$ be a special filtered Kan complex. By a <u>filtered homotopy</u> of $R(\underline{K})$ is meant a simplicial map $f : \Delta(n) x I \longrightarrow R(\underline{K})$ such that :

(a) $f(\Delta(n,r) x I) \subset K^r$ for r = 0,...,n.

Using the notion of filtered homotopy we can define a relation on $R(\underline{K})$ as follows. Let $x,y \varepsilon R(\underline{K})_n$. Then we say x is <u>filtered homotopic</u> to y, written $x \equiv y$, if there exists a filtered homotopy $f : \Delta(n) x I \longrightarrow R(\underline{K})$ such that $f(d_0 a_0) = x$ and $f(d_{n+1} a_n) = y$. We say that f is a filtered homotopy between x and y and we write $f : x \equiv y$.

Proposition 2.7.

Let $\underline{K}$ be a special filtered Kan complex. Then the relation of filtered homotopy on $R(\underline{K})$ is an equivalence relation.

Proof. Let $x \in R(\underline{K})_n$. Then we define a simplicial map $e_x : \Delta(n) x I \longrightarrow R(\underline{K})$ by $e_x(a_i) = s_i x$ for $i = 0,\ldots,n$. Then $e_x : x \equiv x$ and we call e_x the identity filtered homotopy on x.

For symmetry and transitivity it suffices to show that for all $x,y,z \in R(\underline{K})_n$ $x \equiv y$ and $x \equiv z$ implies $y \equiv z$. Suppose $x,y,z \in R(\underline{K})_n$ such that $f_0 : x \equiv y$ and $f_1 : x \equiv z$. Letting $D(n)$ be the subcomplex $\Delta(n) x (0) x I \cup \Delta(n) x \dot{I} x I$ we can define a simplicial map $h : D(n) \longrightarrow K^n$ such that :

$h(\Delta(n) x (0) x I) = e_x$ and $h(\Delta(n) x I x (i)) = f_i$ for $i = 0,1$.

The method of proof will consist of extending h to a simplicial map $\hat{h} : \Delta(n) x I^2 \longrightarrow K^n$ in such a way that $\hat{h}(\Delta(n) x (1) x I)$ provides a filtered homotopy between y and z. We need the following technical result.

Lemma A.

For $n \geq 0$ there exists a set $M(K^n)$ of simplicial maps $f : \Delta(n) x I^2 \longrightarrow K^n$ and satisfying :

M.1 $f(\Delta(n) x (0) x I)$ is the identity filtered homotopy.

M.2 $f(\Delta(n) x I x (i))$ is a filtered homotopy for $i = 0,1$.

M.3 $f((d_i \delta^n) x I^2) \in M(K^{n-1})$ for $i = 0,\ldots,n$.

We prove lemma A by means of lemma B.

Lemma B.

Let $g : D(n) \longrightarrow K^n$ be a simplicial map such that g satisfies, up to inclusion, M.1, M.2 and M.3. Then g extends to a simplicial map $\hat{g} : \Delta(n) x I^2 \longrightarrow K^n$ such that $\hat{g} \in M(K^n)$.

Proof. We use induction. We suppose that for $0 \leq i \leq n-1$ every simplicial map

of the form $g : D(i) \longrightarrow K^i$ or $g : D(i)(\bigcup_{0\le j\le k} ((d_j\delta^n)xI^2)) \longrightarrow K^i$ for $0\le k\le i$ such that g satisfies, up to inclusion, M.1,M.2,M.3, extends to a simplicial map $\hat{g} : \Delta(i)xI^2 \longrightarrow K^i$ with $\hat{g}\varepsilon M(K^i)$. Now let $g : D(n) \longrightarrow K^n$ be a simplicial map such that g satisfies, up to inclusion, M.1,M.2 and M.3. We will first extend g over $(d_i\delta^n)xI^2$ for $i = 0,\ldots,n$ as follows. Letting $D_0(n-1)$ be $D(n-1) \cap (d_0\delta^n)xI^2$ we see that $g \mid (d_0\delta^n)xI^2$ is defined on $D_0(n-1)$. We now show that $g \mid (d_0\delta^n)xI^2$ satisfies, up to inclusion, M.1,M.2 and M.3. Since $g(\Delta(n)xIx(i))$ is a filtered homotopy for $i = 0,1$ we have, by 2.6(a), that $g \mid (d_0\delta^n)xIx(i)$ is a filtered homotopy for $i = 0,1$. Furthermore since $g(\Delta(n)x(0)xI)$ is the identity filtered homotopy it is clear that $g((d_0\delta^n)x(0)xI)$ is the identity filtered homotopy. Hence by our inductive hypothesis g extends over $D(n) \cup (d_0\delta^n)xI^2$ such that $g((d_0\delta^n)xI^2)\varepsilon M(K^{n-1})$. By a similiar argu ment g extends over $(D(n)\bigcup_{i=0}^{n}((d_i\delta^n)xI^2)) = A^0(n)$ where $A^0(n)$ is defined as in 2.2, such that $g((d_i\delta^n)xI^2)\varepsilon M(K^{n-1})$ for $i = 0,\ldots,n$. But by lemma 2.2 $\Delta(n)xI^2 \searrow A^0(n)$ and so g extends to a simplicial map $\hat{g} : \Delta(n)xI^2 \longrightarrow K^n$ such that $\hat{g}\varepsilon M(K^n)$..

Finally it is clear that the lemma is true for the case $n = 0$.

This completes the proof of lemma B and so of lemma A.

Corollary to Lemma A.

If $f\varepsilon M(K^n)$, then $f(\Delta(n)x(1)xI)$ is a filtered homotopy.

The proof is straightforward by induction and so will be omitted.

Now returning to our simplicial map $h : D(n) \longrightarrow K^n$, it is clear that h satisfies the conditions of lemma B. Hence h extends to a simplicial map $\hat{h} : \Delta(n)xI^2 \longrightarrow K^n$ such that $\hat{h}\varepsilon M(K^n)$. Then by the corollary to lemma A,

we have that $\hat{h}(\Delta(n)x(1)xI) \longrightarrow K^n$ provides a filtered homotopy between y and z. This completes the proof of proposition 2.7.

We let [x] denote the equivalence class containing x and we let $\rho(\underline{K})_n$ denote the set of equivalence classes of dimension n. Then $\rho(\underline{K}) = (\rho(\underline{K})_n)_{n\geq 0}$ is a simplicial set.

Returning to example 2.5, if $\underline{X}$ is a filtered space it is easy to see that $\rho(\underline{X}) = \rho(K(\underline{X}))$.

We will now prove that if $\underline{K}$ is a special filtered Kan complex, then $\rho(\underline{K})$ is a T-complex. The main work in proving this result is to show that the projection map $p : R(\underline{K}) \longrightarrow \rho(\underline{K})$ is a Kan fibration. In order to prove that $p : R(\underline{K}) \longrightarrow \rho(\underline{K})$ is a Kan fibration we need the following result.

Proposition 2.8.

Let $\underline{K}$ be a special filtered Kan complex. Let $x_0, x_1 \varepsilon R(\underline{K})_n$ with $x_0 \equiv x_1$. Let $f : \wedge^k(n)xI \longrightarrow R(\underline{K})$ be a simplicial map such that $d_i f$ provides a filtered homotopy between $d_i x_0$ and $d_i x_1$ for $i \neq k$. Then f extends to a simplicial map $\hat{f} : \Delta(n)xI \longrightarrow R(\underline{K})$ such that $\hat{f} : x_0 \equiv x_1$.

Proof. Let $x_0, x_1 \varepsilon R(\underline{K})_n$ with $h : x_0 \equiv x_1$. Let $f : \wedge^k(n)xI \longrightarrow R(\underline{K})$ be a simplicial map such that $d_i f$ provides a filtered homotopy between $d_i x_0$ and $d_i x_1$ for $i \neq k$. We first consider the case k = 0. Let E(n) be the subcomplex $\Delta(n)xIx(0) \cup \dot{\Delta}(n)xIxI \cup \wedge^0(n)xIx(1) \cup (v_0)xI^2$ where $v_0 = d_1 \ldots d_n \delta^n$. Then we can define a simplicial map $\hat{f} : E(n) \longrightarrow K^n$ such that :

(a) $\hat{f}(\Delta(n)xIx(0)) = h$

(b) $\hat{f}(\wedge^0(n)xIx(1)) = f$

(c) $\hat{f}(\Delta(n)x(i)xI) = e_{x_i}$ for i = 0,1

(d) $\hat{f} \mid (v_0)xI^2$ is given by an extension of the above maps using condition (b) of definition 2.4 such that $\hat{f} \mid (v_0)xI^2 \subset K^1$.

The method of proof will consist of extending $\hat{f}$ to a simplicial map $\hat{\hat{f}} : \Delta(n)xI^2 \longrightarrow K^n$ in such a way that $\hat{\hat{f}}(\Delta(n)xIx(1))$ provides a filtered homotopy between x_0 and x_1. We need the following technical result. Recall that we write $\dot{I}^2$ for $IxI \cup \dot{I}xI$.

Lemma C.

Let $f : \Delta(n)x\dot{I}^2 \cup (v_0)xI^2 \longrightarrow K^n$ or

$f : \Delta(n)x\dot{I}^2 \cup (v_0)xI^2 \underset{n\geq j\geq k}{\cup} (d_j\delta^n)xI^2 \longrightarrow K^n$ for $n\geq k\geq 1$

be a simplicial map such that :

(a) $f(\Delta(n)x(i)xI)$ is the identity filtered homotopy for $i = 0,1$.

(b) $f(\Delta(n)xIx(i))$ is a filtered homotopy for $i = 0,1$.

(c) $f \mid (v_0)xI^2 \subset K^1$.

Then f extends to a simplicial map $\hat{f} : \Delta(n)xI^2 \longrightarrow K^n$.

Proof. We use induction. We assume that for $1\leq i\leq n-1$ every simplicial map of the form $f : \Delta(i)x\dot{I}^2 \cup (v_0)xI^2 \longrightarrow K^i$ or

$f : \Delta(i)x\dot{I}^2 \cup (v_0)xI^2 \underset{i\geq j\geq k}{\cup} (d_j\delta^i)xI^2 \longrightarrow K^i$ for $i\geq k\geq 1$, which satisfies the hypothesis of lemma C, extends to a simplicial map $\hat{f} : \Delta(i)xI^2 \longrightarrow K^i$.

Now let $f : \Delta(n)x\dot{I}^2 \cup (v_0)xI^2 \longrightarrow K^n$ be a simplicial map such that f satisfies the hypothesis of the lemma. We will first extend f over $(d_n\delta^n)xI^2$. We see that $f \mid (d_n\delta^n)xI^2$ is defined on $\Delta(n-1)x\dot{I}^2 \cup (v_0)xI^2$. We now show that $f \mid (d_n\delta^n)xI^2$ satisfies the hypothesis of the lemma. Since $f(\Delta(n)xIx(i))$ is a filtered homotopy for $i = 0,1$ we have, by 2.6(a), that $f \mid ((d_n\delta^n)xIx(i))$ is a filtered homotopy for $i = 0,1$. Furthermore since $f(\Delta(n)x(i)xI)$ is the identity filtered homotopy for $i = 0,1$ it is clear that $f \mid ((d_n\delta^n)x(i)xI)$

is the identity filtered homotopy for i = 0,1. Hence by our inductive hypothesis f extends over $(d_n\delta^n)xI^2$ such that $f((d_n\delta^n)xI^2) \subset K^{n-1}$. By a similiar argument f extends over $\Delta(n)x\dot{I}^2 \cup (v_0)xI^2 \bigcup_{n\geq i\geq 1} (d_i\delta^n)xI^2 = C^0(n)$ where $C^0(n)$ is defined as in 2.2, such that $f((d_i\delta^n)xI^2) \subset K^{n-1}$ for i = 1,...,n But by lemma 2.2 $\Delta(n)xI^2 \searrow C^0(n)$ and so f extends to a simplicial map $\hat{f} : \Delta(n)xI^2 \longrightarrow K^n$.

Finally by condition (b) of definition 2.4 the lemma is true for the case n = 1. This completes the proof of lemma C.

We now return to our simplicial map $\hat{f} : E(n) \longrightarrow K^n$ which we defined earlier. By lemma C we can extend $\hat{f}$ over $E(n) \bigcup_{n\geq i\geq 1} (d_i\delta^n)xI^2$ such that $\hat{f}((d_i\delta^n)xI^2) \subset K^{n-1}$ for $n\geq i\geq 1$. Note that this is possible since by 2.6(a) $\hat{f} \mid (d_i\delta^n)xIx(j) \subset K^{n-1}$ for $n\geq i\geq 1$, j = 0,1.

We now extend $\hat{f}$ over $(d_0\delta^n)xI^2$. But $\hat{f} \mid ((d_0\delta^n)xI^2)$ is defined over $B^0(n-1)$, where $B^0(n-1)$ is defined as in 2.2, with $\hat{f} \mid B^0(n-1) \subset K^{n-1}$. (This follows from the fact that $f((d_i\delta^n)xI^2) \subset K^{n-1}$ for $i\neq 0$, and that $\hat{f}((d_0\delta^n)xIx(0)) \subset K^{n-1}$ by condition (a) of definition 2.6) Hence by 2.2 $\hat{f}$ extends over $(d_0\delta^n)xI^2$ such that $\hat{f}((d_0\delta^n)xI^2) \subset K^{n-1}$. Thus $\hat{f}$ is defined over $B^0(n)$ and so by lemma 2.2 $\hat{f}$ extends to a simplicial map $\hat{\hat{f}} : \Delta(n)xI^2 \longrightarrow K^n$. It now follows that $\hat{\hat{f}}(\Delta(n)x(1)xI)$ provides a filtered homotopy between x_0 and x_1 and is an extension of f.

The cases $0<k\leq n$ are similiar to the case k = 0 and so will be omitted. This completes the proof of proposition 2.8.

Using a simple inductive argument we now state the following corollary to proposition 2.8.

Corollary 2.9.

Let $\underline{K}$ be a special filtered Kan complex. Let $x,y \varepsilon R(\underline{K})_n$ with $x \equiv y$. Let λ be a proper subset of the set $\{0,\ldots,n\}$ and suppose $f : \bar{\wedge}^{\lambda}(n) x I \longrightarrow R(\underline{K})$ is a simplicial map such that $d_i f$ provides a filtered homotopy between $d_i x$ and $d_i y$ for $i \varepsilon \lambda$. Then f extends to a simplicial map $\hat{f} : \Delta(n) x I \longrightarrow R(\underline{K})$ such that $\hat{f} : x \equiv y$.

Theorem 2.10.

Let $\underline{K}$ be a special filtered Kan complex. Then the projection map $p : R(\underline{K}) \longrightarrow \rho(\underline{K})$ is a Kan fibration.

We shall prove theorem 2.10 by proving theorem 2.11.

Let $\underline{K}$ be a special filtered Kan complex. Then we define a graded subset $T^1(\underline{K}) = (T^1(\underline{K})_n)_{n \geq 1}$ of $R(\underline{K})$ by $T^1(\underline{K})_n = K_n^{n-1}$ for $n \geq 1$. The image of $T^1(\underline{K})_n$ in $\rho(\underline{K})_n$ is denoted by $T(\underline{K})_n$. Then $T(\underline{K}) = (T(\underline{K})_n)_{n \geq 1}$ is a graded subset of $\rho(\underline{K})$ and elements of $T(\underline{K})$ are called thin.

Theorem 2.11.

Let $\underline{K}$ be a special filtered Kan complex. Let λ be a proper subset of the set $\{0,\ldots,n\}$. Suppose given a commutative diagram

$$\begin{array}{ccc} \bar{\wedge}^{\lambda}(n) & \xrightarrow{\quad f \quad} & R(\underline{K}) \\ {\scriptstyle i}\downarrow & & \downarrow{\scriptstyle p} \\ \Delta(n) & \xrightarrow[\quad g \quad]{} & \rho(\underline{K}) \end{array}$$

then there is an $\hat{f} : \Delta(n) \longrightarrow R(\underline{K})$ such that $\hat{f}i = f$ and $p\hat{f} = g$. Further if $g(\delta^n)$ is thin then $\hat{f}$ may be chosen such that $\hat{f}(\delta^n) \varepsilon T^1(\underline{K})_n$.

Proof. Let $f : \bar{\Lambda}^{\lambda}(n) \longrightarrow R(\underline{K})$ and $g : \Delta(n) \longrightarrow \rho(\underline{K})$ with $pf = g \mid \bar{\Lambda}^{\lambda}(n)$, where λ is a proper subset of the set $\{0,\ldots,n\}$. Let $g(\delta^n) = [x] \varepsilon \rho(\underline{K})_n$, and let $y \varepsilon R(\underline{K})_n$ be a representative of $[x]$. (If $[x] \varepsilon T(\underline{K})_n$ then we may choose $y \varepsilon K_n^{n-1}$.) Then letting $f(d_i \delta^n) = v_i \varepsilon R(\underline{K})_{n-1}$ for $i \varepsilon \lambda$, we see that v_i is filtered homotopic to $d_i y$ for $i \varepsilon \lambda$. Now by successively applying corollary 2.9 we can define a simplicial map $h : \bar{\Lambda}^{\lambda}(n) x I \cup \Delta(n) x (1) \longrightarrow K^n$ (for the case $[x] \varepsilon T(\underline{K})_n$ we may define $h : \bar{\Lambda}^{\lambda}(n) x I \cup \Delta(n) x (1) \longrightarrow K^{n-1}$) such that $h \mid (d_i \delta^n) x I$ provides a filtered homotopy between v_i and $d_i y$ for $i \varepsilon \lambda$ and $h(\Delta(n) x (1)) = y$. It is now clear that using a simple inductive arguement h extends to a simplicial map $\hat{h} : \Lambda^k(n) x I \cup \Delta(n) x (1) \longrightarrow K^n$ ($\hat{h} : \Lambda^k(n) x I \cup \Delta(n) x (1) \longrightarrow K^{n-1}$ in the case when $[x] \varepsilon T(\underline{K})_n$) for k not a member of λ , such that $h \mid (d_i \delta^n) x I$ is a filtered homotopy for $i \neq k$. Now applying lemma 2.1 twice, we see that $\hat{h}$ extends to a simplicial map $\hat{\hat{h}} : \Delta(n) x I \longrightarrow K^n$ ($\hat{\hat{h}} : \Delta(n) x I \longrightarrow K^{n-1}$ in the case when $[x] \varepsilon T(\underline{K})_n$) such that $\hat{\hat{h}}((d_k \delta^n) x I) \subset K^{n-1}$. We now define $\hat{f} : \Delta(n) \longrightarrow R(\underline{K})$ by $\hat{f}(\delta^n) = \hat{\hat{h}}(\delta^n, 0)$. Then $p\hat{f} = g$ and $\hat{f}$ is an extension of f. Further if $g(\delta^n)$ is thin then $\hat{f}$ may be chosen such that $\hat{f}(\delta^n) \varepsilon K_n^{n-1}$. This completes the proof of theorem 2.11 and so of theorem 2.10.

We now use theorem 2.11 to prove a technical result which will enable us to prove that for a special filtered Kan complex $\underline{K}$, $(\rho(\underline{K}), T(\underline{K}))$ is a T-complex.

Proposition 2.12.

Let $\underline{K}$ be a special filtered Kan complex. Suppose $([x_0],\ldots,[x_{i-1}],-,[x_{i+1}],\ldots,[x_n])$ constitutes a box in $\rho(\underline{K})_{n-1}$. Then we can choose representatives $y_j \varepsilon R(\underline{K})_{n-1}$ for $[x_j]$ $j \neq i$ such that $(y_0,\ldots,y_{i-1},-,y_{i+1},\ldots,y_n)$ is a box in $R(K)_{n-1}$. Further if $[x_j] \varepsilon T(\underline{K})_{n-1}$ $j \neq i$ then y_j may be chosen to be in K_{n-1}^{n-2}.

Proof. Let $([x_0],\ldots,[x_{i-1}],-,[x_{i+1}],\ldots,[x_n])$ constitute a box in $\rho(\underline{K})_{n-1}$. Since $p : R(\underline{K}) \longrightarrow \rho(\underline{K})$ is onto we can pick an element $y_0 \varepsilon R(K)_{n-1}$ such that $p(y_0) = [x_0]$. Further if $[x_0]\varepsilon T(\underline{K})_{n-1}$ then y_0 can be chosen to be in K^{n-2}_{n-1}. By theorem 2.11 we can pick an element $y_1 \varepsilon R(\underline{K})_{n-1}$ such that $p(y_1) = [x_1]$ and $d_0 y_1 = d_0 y_0$. Further if $[x_1]\varepsilon T(\underline{K})_{n-1}$ then y_1 can be chosen to be in K^{n-2}_{n-1}. By successive application of theorem 2.11 we can pick an element $y_i \varepsilon R(\underline{K})_{n-1}$ for $j \neq i$, successively so that $p(y_j) = [x_j]$ and $(y_0,\ldots,y_{i-1},-,y_{i+1},\ldots,y_n)$ constitutes a box in $R(\underline{K})_{n-1}$. Further if $[x_j]\varepsilon T(\underline{K})_{n-1}$ then the y_j may be chosen to be in K^{n-2}_{n-1} for $j \neq i$. This completes the proof of 2.12.

Theorem 2.13.

Let $\underline{K}$ be a special filtered Kan complex. Then $(\rho(\underline{K}),T(\underline{K}))$ is a T-complex.

Proof. We show that $(\rho(\underline{K}),T(\underline{K}))$ satisfies the three axioms for a T-complex. The verification of T.1 is trivial since if $x\varepsilon R(\underline{K})_{n-1}$ then $s_i x \varepsilon K^{n-1}_n$. To verify T.2 we see that by proposition 2.12 every box in $\rho(\underline{K})$ has a thin filler. Thus for T.2 we only need uniqueness. Suppose $[x_0],[x_1]\varepsilon T(\underline{K})_n$ are such that $d_i[x_0] = d_i[x_1]$ for $i \neq k$. We need to prove $[x_0] = [x_1]$. Let $y_0, y_1 \varepsilon K^{n-1}_n$ be representatives of $[x_0],[x_1]$ respectively. Then it is sufficient to show that $y_0 \equiv y_1$. By corollary 2.9 we can form a simplicial map $f : \wedge^k(n)xI \cup \Delta(n)x\dot{I} \longrightarrow K^{n-1}$ such that $f(\Delta(n)x(i)) = y_i$ for $i = 0,1$ and $d_j f$ provides a filtered homotopy between $d_j y_0$ and $d_j y_1$ for $j \neq k$. Then it is clear that f extends to a simplicial map $\hat{f} : \Delta(n)xI \longrightarrow K^{n-1}$. Hence $\hat{f} : y_0 \equiv y_1$ as required. To verify T.3 let $[x]\varepsilon T(\underline{K})_n$ such that $d_i[x]\varepsilon T(\underline{K})_{n-1}$ for $i \neq k$. Then we must show that $d_k[x]\varepsilon T(\underline{K})_{n-1}$. By 2.12 we can choose representatives $y_i \varepsilon K^{n-2}_{n-1}$ for $d_i[x]$ when $i \neq k$ such that $(y_0,\ldots,y_{k-1},-,y_{k+1},\ldots,y_n)$ is a box in K^{n-2}_{n-1}. Letting $y\varepsilon K^{n-2}_n$ be a filler of this box we have that $d_k p(y) = p(d_k y)\varepsilon T(\underline{K})_{n-1}$ and $d_i p(y) = d_i[x]$ for $i \neq k$ as required. This completes the proof.

§3. The crossed complex associated to the T-complex $\rho(\underline{X})$.

Let $\underline{X} : X_0 \subset X_1 \subset \ldots \subset X_n \subset \ldots$ be a filtered space satisfying the condition that loops in X_0 are contractible in X_1. Recall that from example 2.5 , using $\underline{X}$ we constructed a special filtered Kan complex $K(\underline{X})$ and a simplicial set $\rho(\underline{X})$ which is easily seen to be $\rho(K(\underline{X}))$. Thus by theorem 2.13 $\rho(\underline{X})$ is a T-complex.

Recall that from chapter 1 section 3, for $n\geq 2$ and $a\varepsilon X_0$ $\rho(\underline{X})^n_a(A) = \{[x]\varepsilon\rho(\underline{X}) : d_i[x] = [s_0^{n-1}a] \text{ for } 1\leq i\leq n\}$ can be given a group structure.

In this section we prove there is a natural bijection between $\rho(\underline{X})^n_a(A)$ and the usual relative homotopy group $\pi_n(X_n,X_{n-1},a)$ for $a\varepsilon X_0$.

For convenience we write $\dot{\Delta}^n$ for $\Delta^{n,n-1}$ (see example 2.5) and J_0^n for $\dot{\Delta}^n\backslash d_0\Delta^n$.

Theorem 3.1.

Let $\underline{X} : X_0 \subset X_1 \subset \ldots \subset X_n \subset \ldots$ be a filtered space satisfying the condition that loops in X_0 are contractible in X_1. Then for $n\geq 2$ and $a\varepsilon X_0$ there is a natural bijection between $\rho(\underline{X})^n_a(A)$ and $\pi_n(X_n,X_{n-1},a)$.

Proof. Let $[x]\varepsilon\rho(\underline{X})^n_a(A)$. We first show that we can pick a representative $y\varepsilon R(K(\underline{X}))_n$ of $[x]$ such that $y : (\Delta^n,\dot{\Delta}^n,J_0^n)\longrightarrow(X_n,X_{n-1},a)$. Using theorem 2.11 we have the required result. Now to prove the theorem it is sufficient to show that if $a\varepsilon X_0$ and $y_0,y_1\varepsilon R(K(\underline{X}))_n$ such that :

(a) $y_0\equiv y_1$

(b) $d_iy_0 = d_iy_1 = s_0^{n-1}a$ for $1\leq i\leq n$

then we can choose a filtered homotopy $h : y_0\equiv y_1$ such that d_ih is the identity

filtered homotopy for $1 \leq i \leq n$. Using proposition 2.8 we have the required result. This completes the proof of theorem 3.1.

CHAPTER 3

Simplicial Groups

In this chapter we shall apply the previous theory to the category of simplicial groups. We define the notion of a group T—complex in the category of simplicial groups and obtain an explicit formula for determining when a simplical group is a group T—complex. Following this we show that every T—complex contains a group T—complex. This leads us to define a new category whose objects are certain group T—complexes over a groupoid. We then prove that this new category is equivalent to the category of T—complexes. Finally we construct a filtered Kan complex from a simplicial group.

§1. Group T—complexes.

Recall that a simplicial set $((G_n)_{n\geq 0}, d_i, s_i)$ is called a simplicial group if each G_n is a group, all d_i, s_i are homomorphisms. A product of degenerate elements in G_n is called slim.

Throughout this section we adhere to the following notation. Let $G = ((G_n)_{n\geq 0}, d_i, s_i)$ be a simplicial group. Then for $n\geq 0$ and $j = 0,\dots,n$ we let $A_n^j = \bigcap_{i\neq j}$ (kernel $d_i : G_n \longrightarrow G_{n-1}$), $A_n = \bigcup_{j=0}^{n} A_n^j$ and $A = (A_n)_{n\geq 1}$. We let $D = (D_n)_{n\geq 1}$ be the graded subgroup of G generated by the degenerate elements. We let 1_n denote the identity of G_n and e the complex consisting of all the 1_n. Where no confusion arises we write 1 for 1_n.

It is well known that every simplicial group is a Kan complex in the following way.

Proposition 1.1 [cf. May p.67]

Let $G = (G_n)_{n\geq 0}$ be a simplicial group. Then every box in G has a filler in D.

Proof. Let $(y_0,\ldots,y_{k-1},-,y_{k+1},\ldots,y_n)$ constitute a box in G_{n-1}. There are three cases.

(1) k=0. Let $w_n = s_{n-1}y_n$ and $w_i = w_{i+1}(s_{i-1}d_iw_{i+1})^{-1}s_{i-1}y_i \quad i = n,\ldots,1.$

Then $w_1 \varepsilon D_n$ and satisfies $d_iw_1 = y_i \quad i\neq 0$.

(2) 0<k<n. Let $w_0 = s_0y_0$ and $w_i = w_{i-1}(s_id_iw_{i-1})^{-1}s_iy_i \quad i = 0,\ldots,k-1.$

$$w_n = w_{k-1}(s_{n-1}d_nw_{k-1})^{-1}s_{n-1}y_n$$

$$w_i = w_{i+1}(s_{i-1}d_iw_{i+1})^{-1}s_{i-1}y_i \quad i = n,\ldots,k+1.$$

Then $w_{k+1}\varepsilon D_n$ and satisfies $d_iw_{k+1} = y_i \quad i\neq k$.

(3) k=n. Let $w_0 = s_0y_0$ and $w_i = w_{i-1}(s_id_iw_{i-1})^{-1}s_iy_i \quad i = 0,\ldots,n-1.$

Then $w_{n-1}\varepsilon D_n$ and satisfies $d_iw_{n-1} = y_i \quad i\neq n$.

We call w_k the standard slim filler.

Definition.

By a group T-complex is meant a T-complex (G,T) where G is a simplicial group and T is a graded subgroup of G.

Proposition 1.2.

If (G,T) is a group T-complex, then T = D.

Proof. By axiom T.1 of a T-complex and the fact that T is a graded subgroup we have that $D\subset T$. Conversely, let $t\varepsilon T_n$. Then the box $(-,t_1,\ldots,t_n)$ has a filler $d\varepsilon D$ by proposition 1.1. Then $d\varepsilon T_n$ also, and so by the uniqueness part of axiom T.2, d = t. Hence $t\varepsilon D$.

Theorem 1.3

Let G be a simplicial group. Then (G,D) is a T-complex if and only if $A\cap D = e$.

Proof. For if (G,D) is a T-complex, then any box of the form $(1,\ldots,1,-,1,\ldots,1)$ has a unique thin filler, namely 1. Conversely suppose

$A\cap D = e$. Then we must show that (G,D) satisfies axioms T.2 and T.3 of a T-complex. To verify T.2 we see that by 1.1 every box in (G,D) has a standard degenerate filler. Thus for T.2 we only need uniqueness. Suppose $(y_0,\ldots,y_{k-1},-,y_{k+1},\ldots,y_n)$ is a box in G_{n-1} with a filler $y\varepsilon D_n$. Then we must show that y is the standard slim filler. There are three cases.

(1) $k = 0$. Let $y^n = y^{-1}s_{n-1}d_n y$ and $y^i = y^{i+1}(s_{i-1}d_i y^{i+1})^{-1}$ $i = n,\ldots,1$. Then $y^1 \varepsilon D_n$ and a simple calculation shows that y^1 fills the box $(-,1,\ldots,1)$. But we are assuming $A_n \cap D_n = 1$. Hence $y^1 = 1$. Now substuting for y^i $i = n,\ldots,1$ we obtain $1 = y^{-1}w$ where w is the standard slim filler. Hence $y = w$ as required. The cases $0<k<n$ are similiar to the case $k = 0$ and so will be omitted. To verify T.3 let $(d_0,\ldots,d_{k-1},-,d_{k+1},\ldots,d_n)$ constitute a box in D_{n-1}. Then the standard slim filler will be of the form $s_i s_j x_\alpha$ for $x_\alpha \varepsilon G_{n-1}$, which clearly has its shell in D_{n-1}. This completes the proof of theorem 1.3.

<u>Corollary 1.4.</u>

Let G be a simplicial group such that (G,D) is a T-complex. Then for $n\geq 3$ A_n^0 is an abelian subgroup of G_n. Further G_n is isomorphic to $A_n^0 x D_n$.

<u>Proof.</u> Recall that from chapter 1, section 1, we can define a partial law of composition on G as follows. Let $x,y\ \varepsilon G_n$ with $d_{n-1}x = d_n y$. Then we define $x + y = d_n M[x,y]$ where $M[x,y]\ \varepsilon\ D_{n+1}$ fills the box $(M[d_0x,d_0y],\ldots,M[d_{n-2}x,d_{n-2}y],y,-,x)$. Then we have the following :

<u>Lemma A.</u>

For $n\geq 3$ let $x,y\varepsilon A_n^0$. Then $M[x,y] = s_{n-1}ys_nx$.

The proof is straightforward by induction and so will be omitted.

Thus for $n\geq 3$ and $x,y\varepsilon A_n^0$ we have that $x + y = yx$. But from chapter 1, section 3, we have that for $n\geq 3$ $(A_n^0,+)$ is an abelian group. It now follows that for

$n\geq 3$, A_n^0 is an abelian subgroup of G_n.

We now show that for $n\geq 3$, G_n is isomorphic to $A_n^0 \times D_n$. By theorem 1.3 and above it is sufficient to prove that A_n^0 and D_n generate G_n. This we now do. Let $x\varepsilon G_n$. Then we let $\tilde{x}\varepsilon D_n$ fill the box $(-,x_1^{-1},\ldots,x_n^{-1})$. Note that this box is well defined since x^{-1} has as its shell $(x_0^{-1},\ldots,x_n^{-1})$. It now follows that $x\tilde{x}\varepsilon A_n^o$ and $x = (x\tilde{x})\tilde{x}^{-1}$. This completes the proof.

Recall that a simplicial abelian group is a simplicial group $G = (G_n)_{n\geq 0}$ such that G_n is abelian for $n\geq 0$.

<u>Theorem 1.5</u>

Every simplicial abelian group is a group T-complex.

<u>Proof.</u> Let $G = (G_n)_{n\geq 0}$ be a simplicial abelian group. Then to prove the theorem we must show that (G,D) is a T-complex. By theorem 1.3 it is sufficient to show that $A\cap D = e$. This we now do. Since G_n is abelian, any element in D_n is of the form $s_0x_0\ldots s_{n-1}x_{n-1}$ for $x_i\varepsilon G_{n-1}$. Suppose $d\varepsilon D_n$ and fills the box $(1,\ldots,1,-,1,\ldots,1)$ then we must show that $d=1$. There are various cases.

(1) $k = 0$. Suppose $d = s_0x_0\ldots s_{n-1}x_{n-1}$ fills the box $(-,1,\ldots,1)$. Then $d_nd = (s_0d_{n-1}x_0\ldots s_{n-2}d_{n-1}x_{n-2})x_{n-1} = 1$ and so $x_{n-1}\varepsilon D_{n-1}$. It then follows that $x_{n-2},\ldots,x_0\varepsilon D_{n-1}$. Hence $d = s_0s_0y_0s_0s_1y_1\ldots s_{n-2}s_{n-2}y_j$ where $y_l\varepsilon G_{n-2}$ for $0\leq l\leq j$. We now suppose that $d = s_0^iz_0s_0^{i-1}s_1z_1\ldots s_{n-i-1}^{i-1}s_{n-i}z_{k-1}s_{n-i}^iz_k$ where $z_l\varepsilon G_{n-i}$ $0\leq l\leq k$. Then

$d_{n-i-1}\ldots d_{n-1}d_nd = (s_0^id_{n-i-2}\ldots d_{n-2}d_{n-1}z_0\ldots s_{n-i-1}d_{n-i}z_{k-1})z_k = 1$. Hence $z_k\varepsilon D_{n-i}$. It now follows that $z_{k-1},\ldots,z_0\varepsilon D_{n-i}$. Hence for $i = n$ we see that $d = s_0^nv$. Then $d_1\ldots d_ns_0^nv = 1$ and so $v = 1$ as required. The cases $0<k\leq n$ are similiar to the case $k = 0$ and so will be omitted. This completes the proof.

We now have the following well known result.

Corollary 1.6 [cf. May p.94]

Let G be a simplicial abelian group. Then for $n\geq 1$ G_n is isomorphic to $A^0_n \times D_n$.

We state the following trivial lemma.

Lemma 1.7.

Let $m\geq 0$ and suppose G is a simplicial group such that G_n is abelian for $n\geq m$. Then G is a simplicial abelian group.

The proof follows from the fact that each s_i is an injective homomorphism.

Theorem 1.8.

Let $G = (G_n)_{n\geq 0}$ be a simplicial group such that :

(a) (G,D) is a T-complex.

(b) G_2 is abelian.

Then G is a simplicial abelian group.

Proof. We use induction. We suppose G_i is abelian for $i = 2,\ldots,n-1$. Then we must prove G_n is abelian. By corollary 1.4 it is sufficient to prove that D_n is abelian. This we now do. Let $x,y\in D_n$. Then since we are assuming G_{n-1} is abelian xy and yx fill the same box. Hence by axiom T.2 of a T-complex $xy = yx$ and so G_n is abelian. Now using lemma 1.7 we have the required result.

Theorem 1.9.

If $G = (G_n)_{n\geq 0}$ is a simplicial group such that $G_0 = 1$ and G_1 is non abelian, then (G,D) is not a T-complex.

Proof. Since G_1 is non abelian there exists elements $x,y\in G_1$ such that $xy\neq yx$. Then $s_0x\,s_1y$ and $s_1y\,s_0x$ both fill the box $(x,-,y)$. But $s_0xs_1y\neq s_1ys_0x$ since $d_1(s_0xs_1y) = xy$ whereas $d_1(s_1ys_0x) = yx$. Hence D_2 does not satisfy axiom T.2 of a T-complex. This completes the proof.

To show that non abelian group T-complexes exist consider the following

example. Let H be a non abelian group. Then we can form a simplicial group $\hat{H} = (H_n)_{n\geq 0}$ with $H_n = H$ for all $n\geq 0$, all d_i, s_i, are the identity maps. Then clearly $\hat{H}$ is a group T-complex, the thin elements being H in each dimension $n\geq 1$.

§.2 Special simplicial groups over a groupoid.

We now show that every T-complex contains a group T-complex. Let (K,T) be a T-complex. Recall that from chapter 1 for $a\in K_0$ we constructed groups $K_a^n = \{ x\in K_n : x_{n-1} = x_n = s_0^{n-1}a\}$. It then turned out that for $i = 0,\dots,n-2$ and $a\in K_0$ $d_i : K_a^n \longrightarrow K_a^{n-1}$ and $s_i : K_a^n \longrightarrow K_a^{n+1}$ are homomorphisms. Thus for each $a\in K_0$ we can define a simplicial group $G^a = (G_n^a)_{n\geq 0}$ where $G_n^a = K_a^{n+2}$ $n\geq 0$. We now show that G^a is a group T-complex. Let $S^a = (S_n^a)_{n\geq 1}$ be the graded subset of G^a with $S_n^a = K_a^{n+2} \cap T_{n+2}$ $n\geq 1$. Then it is clear that (G^a,S^a) is a T-complex. Further , by chapter 1, section 3, S^a is a graded subgroup of G^a. Hence (G^a,S^a) is a group T-complex.

The above construction leads us to the following definition.

Definition.2.1.

By a special simplicial group over a groupoid is meant a sequence

$$(G;L_1,L_0) : \dots \longrightarrow G_n \xrightarrow{\delta_n} G_{n-1} \longrightarrow \dots \longrightarrow G_0 \xrightarrow{\delta_0} L_1 \rightrightarrows L_0$$

and satisfying the following axioms.

(1) (L_1,L_0) is a groupoid with objects L_0.

(2) For $n\geq 0$ G_n is a family of groups $(G_n(a))_{a\in L_0}$ and for $a\in L_0$ $G(a) = (G_n(a))_{n\geq 0}$ is a group T-complex such that $A_i^0(a)$ is abelian $i = 0,1$.

(3) For $n\geq 1$ $\delta_n = \prod_{i=0}^{n} d_i^{(-1)^i}$ and $\delta_0 : G_0 \longrightarrow L_1$ is a morphism of groupoids

over L_0.

(4) The groupoid (L_1,L_0) operates to the right on each G_n $n\geq 0$ with an action $(x,p)\longrightarrow x^p$ for $x\varepsilon G_n(a)$, $p\varepsilon L_1(a,b)$. Then $x^p\varepsilon G_n(b)$ and the usual laws hold.

(5) For $n\geq 0$ $\delta_n : G_n \longrightarrow G_{n-1}$ and $\delta_0 : G_0 \longrightarrow L_1$ are morphisms of groupoids over L_0 which preserve the action of L_1, where L_1 operates on the group $L_1(a)$ by conjugation.

(6) If $x\varepsilon G_0$, then $\delta_0 x$ acts trivially on G_n for $n\geq 1$ and operates on G_0 by conjugation by x.

(7) $\delta_0\delta_1$ is trivial.

We remark that for a T-complex (K,T) $(G;L_1,L_0)$ satisfies the formal properties of the sequence

$$\hat{K} : \ldots \longrightarrow (K_a^n)_{a\varepsilon K_0} \longrightarrow (K_a^{n-1})_{a\varepsilon K_0} \longrightarrow \cdots (K_a^2)_{a\varepsilon K_0} \longrightarrow K_1 \rightrightarrows K_0.$$

(see chapter 1, section 4)

Then special simplicial groups over a groupoid are the objects of a category SG in which a morphism $f : (G;L_1,L_0) \longrightarrow (H;M_1,M_0)$ is a family of maps $f_n : G_n \longrightarrow H_n$ for $n\geq 0$ and $\hat{f}_i : L_i \longrightarrow M_i$ $i = 0,1$ compatible with all the groupoid structures, the maps $\delta$and the action of L_1 on G_n.

<u>Theorem 2.2.</u>

The categories T and SG are equivalent.

<u>Proof.</u> We define a functor $M : T \longrightarrow SG$ by $M(K,T) = \hat{K}$ (see above), and we define a functor $E : SG \longrightarrow T$ as follows. Let $(G;L_1,L_0)$ be an object in SG. Then it is clear, from the axioms for a special simplicial groupoid, that $(G;L_1,L_0)$ contains a crossed complex

$$c : \ldots \longrightarrow A_n^0 \longrightarrow \ldots \longrightarrow A_2^0 \longrightarrow A_1^0 \longrightarrow G_0 \longrightarrow L_1 \rightrightarrows L_0. \quad (A_i^0 = \textstyle\bigcup_{a\varepsilon L_0} A_i^0(a))$$

We now let $E(G;L_1,L_0) = Dc$ where D is our functor from the category of

crossed complexes to the category of T-complexes as defined in chapter 1, section 11. It now follows from the theory of chapter 1 that the functors E and M define an equivalence of categories.

§3. A filtration of a simplicial group.

Let $G = (G_n)_{n\geq 0}$ be a simplicial group. Then for $i\geq 0$ we define $G^i = (G^i_n)_{n\geq 0}$ to be the simplicial group such that :

$$G^i_n = \begin{cases} G_n & n = 0,\dots,i. \\ D_{i+1} & n = i+1 \\ \text{The subgroup of } D_n \text{ generated by the elements } s_l x \text{ for } 0\leq l\leq n-1,\ x\in G^i_{n-1}, \\ \text{and } n>i+1. \end{cases}$$

Then it is clear that $G^0 \subset G^1 \subset \dots$ is a special filtered Kan complex.

CHAPTER 4

Miscellaneous

In this chapter we prove some miscellaneous results about T-complexes. Recall that a T-complex of rank n is a T-complex having only thin elements above level n.

Let (J,S) and (K,T) be T-complexes of rank n, and suppose we are given maps $f_i : J_i \longrightarrow K_i$ for $0 \le i \le n$ which commute with the face maps and send thin elements to thin elements. Dakin [1] poses the problem : can the maps f_i for $0 \le i \le n$ be extended to a morphism $f : (J,S) \longrightarrow (K,T)$ of T-complexes. In general the answer is no as can be seen from the following arguement.For if $f : (J,S) \longrightarrow (K,T)$ was a morphism of T-complexes which extended the f_i, then for $x,y \in J_n$ with $x_{n-1} = y_n$ we have that $f_n(xy) = f_n x f_n y$ (see chapter 1,section 7.8). However in our problem we have only stipulated that the f_i commute with the face maps and send thin elements to thin elements. Thus we are lead to the following proposition.

Let (K,T) be a T-complex. Then for $n \ge 1$ we define $M_{n+1}[T] = \{x \in T_{n+1} : x = M[y,z]$ where $y,z \in K_n$ with $y_{n-1} = z_n\}$ ($M[y,z]$ is defined as in chapter 1,section 1).

Proposition 1.

Suppose (J,S) and (K,T) are T-complexes of rank n and $f_i : J_i \longrightarrow K_i$ for $0 \le i \le n$, $f_{n+1} : M_{n+1}[S] \longrightarrow M_{n+1}[T]$ are maps which commute with the face maps and send thin elements to thin elements. Then these maps extend to a unique morphism $f : (J,S) \longrightarrow (K,T)$ of T-complexes.

Proof. By a result of Dakin (see [1] chapter 1, section 3, theorem 3.1) it is sufficient to extend f_{n+1} over S_{n+1}. Suppose $x \varepsilon S_{n+1}$, then we define $f_{n+1}x$ to be the thin filler of the box $(-, f_n d_1 x, \ldots, f_n d_{n+1} x)$. Now to complete the proof we need to show that $d_0 f_{n+1} x = f_n d_0 x$. By lemma 12.5 of chapter 1 we may assume (J,S) is of the form (Dc,Tc) and (K,T) is of the form $(D\bar{c}, T\bar{c})$ where c and $\bar{c}$ are crossed complexes . Then we have that x is of the form $(\underline{c}_0(y_0), \ldots, \underline{c}_{n+1}(y_{n+1}); 1)$ where $y_0, \ldots, y_{n+1} \varepsilon c_n$ Hence by chapter 1 section 11 we have that

$$y_0^{p^{-1}} \prod_{i=1}^{n+1} y_i^{(-1)^i} = 1 \text{ where } p = d_2 \ldots d_n \underline{c}_{n+1}(y_{n+1}).$$

Hence by our hypothesis $1 = f_n(y_0^{p^{-1}} \prod_{i=1}^{n+1} y_i^{(-1)^i}) = f_n(y_0^{p^{-1}}) \prod_{i=1}^{n+1} f_n y_i^{(-1)^i}$ ...(a)

Letting $d_0 f_{n+1}(x) = \underline{c}(z)$ and $f_n d_0 x = \underline{c}(z_1)$ where $\underline{c}(z)$ and $\underline{c}(z_1) \varepsilon (D\bar{c}, T\bar{c})_n$ we must show that $\underline{c}(z) = \underline{c}(z_1)$. It is sufficient to show that $z = z_1$, but this follows from (a). This completes the proof.

Let (K,T) be a T-complex. Recall that from chapter 1 section 10 , we defined a map $\mu_n : K_n \longrightarrow K_a^n(A)$.

Proposition 2.

Let (K,T) be a T-complex. Then for $n \geq 2$ and $x, y \varepsilon K_n$ we have that $x = y$ if and only if $d_i x = d_i y$ for $0 \leq i \leq n$ and $\mu_n x = \mu_n y$.

Proof. By lemma 12.5 of chapter 1 we may assume (K,T) = (Dc,Tc) where c is a crossed complex. Suppose $\underline{c}(x) \varepsilon D_n c$ $(n \geq 2)$, then by 12.1 and 12.2 of chapter 1 $\mu_n \underline{c}(x) = \underline{c}(x^p)$ where $p \varepsilon c_1$ and depends only on the faces of $\underline{c}(x)$. The result now follows.

Corollary 3.

Let (K,T) be a T-complex and suppose $x, y \varepsilon K_n$ $(n \geq 3)$ with $x_{n-1} = y_n$, $y_{n-1} = x_n$ such that $d_i(xy) = d_i(yx)$ for $0 \leq i \leq n$. Then $xy = yx$.

Proof. By proposition 2 it is sufficient to show that $\mu_n(xy) = \mu_n(yx)$. But by lemma 10.1(2) and 10.1(3) of chapter 1 we have that

$$\begin{aligned}\mu_n(xy) &= \mu_n x \mu_n y \\ &= \mu_n y \mu_n x \text{ by 3.2 of chapter 1 } (n \geq 3) \\ &= \mu_n(yx) \text{ as required.}\end{aligned}$$

We now define various subcategories of the categories of T-complexes and crossed complexes.

Recall that a simplicial set L is said to be minimal if whenever $x, y \varepsilon L_n$ with $d_i x = d_i y$ for $i \neq k$, then $d_k x = d_k y$.

We let T(M) be the subcategory of T consisting of minimal T-complexes. We let T(F) be the subcategory of T consisting of T-complexes with the extra property that every shell has a filler.

We let C(0) be the subcategory of C consisting of crossed complexes with zero maps. We let C(E) be the subcategory of C consisting of exact crossed complexes.

Theorem 4.

T(M) is equivalent to C(0) and T(F) is equivalent to C(E).
The proof is easily deduced from the fact that T is equivalent to C.

Let K be a simplicial set, and suppose S and T are graded subsets of K such that (K,S) and (K,T) are T-complexes. Then it is clear that N(K,S) and N(K,T) coincide when considered as graded subsets of K, and N is our functor from the category of T-complexes to the category of crossed complexes as defined in chapter 1 section 7. However (K,S) is not isomorphic to (K,T)

in general as can be seen from the following.

Lemma 5.

Let S be a set with a distinguished element e. Then there is a graded set $\bar{D}S = (\bar{D}_i S)_{i\geq 0}$ and a set of maps $(d_0,\ldots,d_i : \bar{D}_i S \longrightarrow \bar{D}_{i-1} S$ such that :

(a) $\bar{D}_0 S = e$

(b) $\bar{D}_1 S = S$

(c) For $i\geq 2$ elements of $\bar{D}_i S$ are of the form $x = (x_0,\ldots,x_i)$ and satisfying :

(1) $x_j \varepsilon \bar{D}_{i-1} S$ for $0\leq j\leq i$

(2) $d_k x_j = d_{j-1} x_k$ for $j>k$

(3) $d_k x = x_k$ for $0\leq k\leq i$.

The proof is straightforward using induction and so will be omitted.

Let G be a group with identity element e. Then we define a graded subset $TG = (T_i G)_{i\geq 1}$ of $\bar{D}G$ such that :

$$T_i G = \left\{ \begin{array}{l} e \text{ for } i = 1 \\ (x_0,x_1,x_2)\varepsilon\bar{D}_2 G \quad : x_2 x_0 = x_1 \text{ for } i = 2 \\ \bar{D}_i G \text{ for } i\geq 3. \end{array} \right\}$$

Theorem 6.

If G is a group, then $(\bar{D}G, TG)$ is a T-complex.

The proof is clear from the definitions.

Let S_4 be the finite set of four elements. Then S_4 can be given two non isomorphic group structures, namely $Z_2 \oplus Z_2$ and Z_4. Hence it is easy to see that $\bar{D}S_4$ can be given two non isomorphic T-complex structures.
Note that if $\bar{D}S_4$ is a T-complex, then $\bar{D}_1 S_4$ is a group since $\bar{D}_0 S_4$ consists of a single element.

It now follows that for any set S there is a bijection between the number of ways $\bar{D}S$ can be made into a T-complex and the number of ways S can be given a group structure.

We now show how to construct a T-complex from a set and a given T-complex. Let (K,T) be a T-complex and S a set. Using (K,T) and S we define a simplicial set ${}^SK = ({}^SK_n)_{n\geq 0}$ such that :

(a) For $n\geq 0$ ${}^SK_n$ is the set of functions $f : S \longrightarrow K_n$

(b) If $f\varepsilon {}^SK_n$ then for $0\leq i\leq n$ we define

$$\bar{d}_i : {}^SK_n \longrightarrow {}^SK_{n-1} \quad \text{and} \quad \bar{s}_i : {}^SK_n \longrightarrow {}^SK_{n+1}$$
$$f \longmapsto d_i f \qquad\qquad f \longmapsto s_i f$$

Then it is clear that ${}^SK = (({}^SK_n)_{n\geq 0}, \bar{d}_i, \bar{s}_i)$ is a simplicial set. We now let ${}^ST = ({}^ST_n)_{n\geq 1}$ be the graded subset of ${}^SK$ such that ${}^ST_n = \{f\varepsilon {}^SK_n : f(S) \subset T_n\}$. We call elements of ST thin functions.

Theorem 7.

Let (K,T) be a T-complex and S a set. Then $({}^SK, {}^ST)$ is a T-complex.

Proof. We prove that $({}^SK, {}^ST)$ satisfies the three axioms of a T-complex. To verify T.1 let $f\varepsilon {}^SK_{n-1}$ and $x\varepsilon S$. Then $s_i f(x) \varepsilon T_n$ since (K,T) is a T-complex. To verify T.2 let $(f_0,\ldots,f_{i-1},-,f_{i+1},\ldots,f_n)$ constitute a box in ${}^SK_{n-1}$. Then we define a function $f : S \longrightarrow K_n$ as follows. Let $x\varepsilon S$, then we define $f(x)$ to be the thin filler of the box $(f_0(x),\ldots,f_{i-1}(x),-,f_{i+1}(x),\ldots,f_n(x))$. Then f is a well defined thin function such that $\bar{d}_j f = f_j$ for $j\neq i$. Further it is clear that f is the unique thin function which fills the above box. Lastly to verify T.3 let $(t_0,\ldots,t_{i-1},-,t_{i+1},\ldots,t_n)$ be a box in ${}^SK_{n-1}$ with $t_j\varepsilon {}^ST_{n-1}$ for $j\neq i$. We let $t\varepsilon {}^ST_n$ be its unique thin filler. Then we must show

that $\bar{d}_i t \varepsilon {}^S T_{n-1}$. Let $x \varepsilon S$, then $t(x) \varepsilon T_n$ and fills the thin box $(t_0(x), \ldots, t_{i-1}(x), -, t_{i+1}(x), \ldots, t_n(x))$. Hence by axiom T.3 of a T-complex $d_i t(x) \varepsilon T_{n-1}$ as required. This completes the proof of theorem 7.

Let $g : S_0 \longrightarrow S_1$ be a map of sets, and (K,T) a T-complex. Then g induces a morphism $g^* : ({}^{S_1}K, {}^{S_1}T) \longrightarrow ({}^{S_0}K, {}^{S_0}T)$ of T-complexes.

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Appendix.

The proof of uniqueness in theorem 9.1 of chapter 1.

Let f , $\bar{f}$: $(J,S) \rightarrow (K,T)$ be two morphisms of T-complexes agreeing on $N(J,S)$. Then we must show that $f = \bar{f}$. We use induction. We assume that $f_i = \bar{f}_i$ for $0 \leq i \leq n-1$. Then we must show that $f_n = \bar{f}_n$. By theorem 9.1 for $u \varepsilon J_{n-1}$ $f^u_n = \bar{f}^u_n$. But for $x \varepsilon J_n$, $s_{n-2}x_n x I[x_{n-1}] \varepsilon J_n^{s_{n-2}d_{n-1}x_n}$ and so we have that $f_n(s_{n-2}x_n x I[x_{n-1}]) = \bar{f}_n(s_{n-2}x_n x I[x_{n-1}])$. Then by 7.8 (i) of chapter 1 it follows that $s_{n-2}f_{n-1}x_n f_n x I[f_{n-1}x_{n-1}] = s_{n-2}\bar{f}_{n-1}x_n \bar{f}_n x I[\bar{f}_{n-1}x_{n-1}]$. Now by our inductive hypothesis and lemma 8.4 we have that $f_n x = \bar{f}_n x$ as required.