An introduction to simplicial T-complexes

By Ronald Brown

Abstract: The notion of *simplicial T-complex* was found by M.K. Dakin in 1975, and was written up in his Ph.D. thesis [D2]. The cubical version of T-complex was taken up by R. Brown and P.J. Higgins in the period 1975-1979, and in particular was found to be an essential feature of their proof of a higher dimensional form of the Seifert-van-Kampen theorem [B-H 2, 3]. Meanwhile, the simplicial T-complex techniques were developed by N.K. Ashley in his Ph.D. thesis [As]; his major result is that conjectured in [D2], the *equivalence between the category of simplicial T-complexes and the category of crossed complexes.*

While the details of the theory and applications of cubical T-complexes have been published [B-H 2, 3, 4], no details are generally available on the simplicial case. It is a pleasure therfore to thank Mme. Andrée Charles Ehresmann for this opportunity to present the Ph.D. Theses of Dakin and Ashley in this issue of Esquisses Mathématiques. I hope it will also prove useful to give in this introduction an account of how the notion of T-complex arose, to summarise the results and methods of the thwo theses, and finally to place this work in the context of algebraic topology, and work on multiple groupoids and higher dimensional forms of the Seifert-van-Kampen theorem.

Keywords: Simplicial sets, Cubical sets, Simplicial T-complex, Kan complex, Seifert-van-Kampen Theorem, Crossed complex, omega-groupoids, Kan fibration, Homotopy, Holomolgy, Classifying space.

Originally published by Esquisses Mathématiques, volume 32, in 1983.

An introduction to simplicial T-complexes

by Ronald Brown

The notion of *simplicial* T-*complex* was found by M.K. Dakin in 1975, and was written up in his Ph.D. thesis [D2]. The cubical version of T-complex was taken up by R. Brown and P.J. Higgins in the period 1975-1979, and in particular was found to be an essential feature of their proof of a higher dimensional form of the Seifert-van Kampen theorem [B-H 2,3]. Meanwhile, the simplicial T-complex techniques were developed by N.K. Ashley in his Ph.D. thesis [As]; his major result is that conjectured in [D2], the *equivalence between the category of simplicial* T-*complexes and the category of crossed complexes*.

While the details of the theory and applications of cubical T-complexes have been published [B-H 2,3,4], no details are generally available on the simplicial case. It is a pleasure therefore to thank Mme. Andrée Charles Ehresmann for this opportunity to present the Ph.D. Theses of Dakin and Ashley in this issue of Esquisses Mathématiques. I hope it will also prove useful to give in this introduction an account of how the notion of T-complex arose, to summarise the results and methods of the two theses, and finally to place this work in the context of algebraic topology, and work on multiple groupoids and higher dimensional forms of the Seifert-van Kampen theorem.

1. Kan complexes

A simplicial T-complex is a special kind of *Kan simplicial set*, i.e. of simplicial set K satisfying Kan's *extension condition*. This condition on K is very well known in simplicial theory, and is often expressed as: *every horn in* K *has a filler*. It will prove useful to give the definition.

Let K be a simplicial set, and let $n \geq 1$, $0 \leq k \leq n$. An (n, k)-*horn* in K is a collection (x_i) of elements x_i of K_{n-1} satisfying $\partial_i x_j = \partial_{j-1} x_i$ for all $i < j$ and $i, j \neq k$. A *filler* of this horn is an element x of K_n such that $\partial_i x = x_i$, for $i \neq k$.

Alternatively, let Δ^n denote the simplicial set of the standard n-simplex (i.e. Δ^n is the free simplicial set on one generator c^n of dimension n). Let Λ^n_k denote the k-*horn* of Δ^n , that is, Λ^n_k is the subsimplicial set generated by all faces $\partial_i c^n$ of c^n for $i \neq k$. Then a horn (x_i) in K can equally be described as a simplicial map $h : \Lambda^n_k \to K$ (with $h(\partial_i c^n) = x_i$, $i \neq k$) and a filler x of (x_i) can be described as an extension $f : \Delta^n \to K$ of h (with $f(c^n) = x)$.

The extension condition was formulated by Kan initially in a cubical context [K1, 2], and shown to be a sufficient condition for doing homotopy theory in a cubical complex K. The reason for this is that the extension condition on K ensures that the relation of homotopy, on maps $L \to K$ of cubical complexes, is an equivalence relation. In later work of Kan [K3, 4], and of many others (see for example [Ma, La, G-Z] and the survey article [C]), this condition was given in the simplicial context (as above) and again shown to be sufficient for homotopy theory of simplicial sets. Indeed, attention has since focussed almost completely on simplicial theory, and many areas of cubical theory have not been developed. For example, there is no complete and widely available account of realisations of cubical sets (the only references I know on this are [Fe, Hin]). From now on, then, by *Kan complex* we mean *Kan simplicial set.*

There are three main examples of Kan complexes.

(1.1) <u>The singular complex</u>

The first, and most intuitive example, is the (simplicial) singular complex SX of a space X. Let us write $|\Delta^n|$, $|\Lambda^n_k|$ for the standard geometric n-simplex and its k-horn, respectively. Then S_nX consists of the continuous maps $|\Delta^n| \to X$, and the fact that SX is a Kan complex is an immediate consequence of the fact that $|\Lambda^n_k|$ is a retract of $|\Delta^n|$.

(1.2) <u>Simplicial groups</u>

Simplicial groups were proved to be Kan complexes by J.C. Moore [Mo]. For a proof, see [Ma, §17]. This result has been useful because many constructions lead from spaces or simplicial sets to simplicial groups, and the latter, being Kan complexes, have a convenient homotopy theory.

(1.3) <u>$Ex^\infty K$</u>

The functor Ex^∞ from simplicial sets to Kan complexes was introduced in [K5]. It has an interesting relation to the notion of subdivision.

A curious fact about these examples, is that the fillers in each case satisfy a naturality condition.

In the case of the singular complex SX, the fillers are defined by retractions on the models $|\Delta^n|$; so they may be chosen on the models once and for all, and the fillers in SX then become natural with respect to maps of X.

In the case of a simplicial group G, a filler x is constructed from a horn (x_i) by the use of the face, degeneracy and group operations of G,

applied to (x_i) . So the filler is natural with respect to maps of simplicial groups.

In the case of $Ex^{\infty}K$, the fillers are natural with respect to maps of K (cf. Lemma 3.2 of [K5]).

The first two examples lead the writer to consider *Kan complexes with canonical fillers*. But such an idea presented three problems.

(1.4) Problem

What relations, if any, should there be between canonical fillers, or in forming successive filling processes?

An awkward feature here is that in the example of the singular complex SX , although the fillers come from retractions $|\Delta^n| \to |\Lambda^n_k|$, such retractions are not unique. However, they are unique up to homotopy.

(1.5) Problem

Can one form a canonical quotient, say σX , of SX so as to obtain a Kan complex in which all fillers coming from retractions on the models have been identified?

Given a solution of (1.5), it would then be more reasonable to look for an answer to (1.4) by using a model category.

(1.6) Problem

Is it possible to define 'Kan complexes with canonical fillers' as contravariant set-valued functors on some model category defined in a combinatorial way?

It is not clear how old these problems are. In any case, by 1975 another line of enquiry had become firmly established. It used cubical theory up to dimension two.

2. The Seifert-van Kampen theorem

In 1965 I had found that the fundamental groupoid could be used to generalise the form of the Seifert-van Kampen theorem given by Crowell in [Cr] from connected to non-connected spaces [B3]. The resulting theorem allowed one to compute the fundamental group of the circle, and many other cases not easily recoverable from the group case, including examples stated in van Kampen's paper [vK]. This discovery led to an extensive use of groupoids in the presentation of the elements of homotopy theory given in [B4]. All this was strongly influenced by work of P.J. Higgins [H1, 2].

By 1967, it had seemed plausible that this theorem should have a generalisation to dimension two. (An account of the argument for this is given

in [B5].) Progress on this problem was negligible until a visit to Bangor in 1972 by Chris Spencer, under S.E.R.C. support. Graeme Segal had mentioned to me in 1971 that he had been told by Deligne that group objects G in the category of groupoids were classified by $H^3(\pi_0 G, G(e, e))$ (where e is the identity of the group $Ob(G)$). Following this hint, Chris and I proved that the category of group objects in the category of groupoids was equivalent to the category of crossed modules (a result in fact known to Verdier in 1965). We then developed a notion of a kind of double groupoid which looked a reasonable candidate for the algebraic gadgets appropriate to a two-dimensional form of the Seifert-van Kampen theorem. This work was issued as a preprint in 1973, and was finally published as [B-S 1,2].

In 1974, Philip Higgins visited Bangor for five months under S.E.R.C. support. Towards the end of his stay, in July, we found the right geometric functor ρ with the above mentioned double groupoids as values. The key was the realisation that crossed modules were obviously relevant, and that they occurred in homotopy theory as the second relative homotopy group $\pi_2(X, Y, z)$ of a based pair, where this group has an operation of $\pi_1(Y, z)$ and a boundary $\partial : \pi_2(X, Y, z) \to \pi_1(Y, z)$. This eventually suggested that ρ *should be defined for a based pair* (X, Y, z) . The simplest idea seemed to be that ρ should consist of :

(0) z in dimension 0 ,

(1) the group $\pi_1(Y, z)$ in dimension 1,

(2) the set of homotopy classes of maps

$$(I^2, \dot{I}^2, \ddot{I}^2) \longrightarrow (X, Y, z)$$

in dimension 2 (where $\dot{I}^2$ and $\ddot{I}^2$ consist respectively of the edges and vertices of I^2).

Of course in dimension two, ρ does not have a group structure. However, it does have *two groupoid structures*. In fact, $\rho(X, Y, z)$ becomes a double groupoid of the form suggested in [B-S 2] (a *special double groupoid with special connection*), and the properties of these allowed a proof of a pushout theorem for ρ . With the work of [B-S 2], this gave a pushout theorem for the second relative homotopy group $\pi_2(X, Y, z)$, considered as a crossed $\pi_1(Y, z)$-module.

Later, the functor ρ was generalised to the case of triples (X, Y, Z) such that each loop in Z is contractible in Y , and the pushout version (when X is the union of two open sets) was generalised to the case of arbitrary open covers. In 1975, these results were circulated as preprints, and were

reported on at Oberwolfach. The preprint influenced work of J. Huebschmann on the cohomology of groups [Hu].

These two-dimensional results were published in [B-H 1], but without the motivation and pictures of the preprint. A recent exposition of the key ideas is given in [B5].

Thus in 1975 the two-dimensional situation was well sorted-out. The problems were about the extension of these results to higher dimensions.

It was clear that the based pairs used to define $\pi_2(X, Y, z)$ and the first version of ρ, should be replaced by a filtered space

$$\underset{\sim}{X} : \{z\} \subset X_1 \subset \ldots \subset X_n \subset X_{n+1} \subset \ldots \subset X .$$

For such an $\underset{\sim}{X}$, a generalisation of the crossed $\pi_1(Y, z)$-module $\pi_2(X, Y, z)$ had been considered by Blakers [Bℓ] (under the name 'group system'), and by J.H.C. Whitehead [W] (under the name 'homotopy system', though with extra freeness assumptions since he was considering the skeletal filtrations of CW-complexes). Eventually, influenced by Huebschmann's 'crossed resolutions' [Hu], we agreed on the name crossed complex for this algebraic gadget which formalised the properties of the relative homotopy groups $\pi_n(X_n, X_{n-1}, z)$, $n \geq 2$, together with the fundamental group $\pi_1(X_1, z)$; the operations of this group on the relative homotopy groups; and the boundary maps

$$\delta : \pi_n(X_n, X_{n-1}, z) \longrightarrow \pi_{n-1}(X_{n-1}, X_{n-2}, z) \quad \text{for} \quad n \geq 3$$

and $\delta : \pi_2(X_2, X_1, z) \longrightarrow \pi_1(X_1, z)$. This structure $\pi\underset{\sim}{X}$ we called the *homotopy crossed complex of the filtered space* $\underset{\sim}{X}$.

Later it became convenient to let the base point z vary in a subset X_0 of X_1, usually with X_0 discrete or with $\pi_0 X_0 = X_0$. Then $\pi_1\underset{\sim}{X}$ became the fundamental groupoid $\pi_1(X_1, X_0)$, and for $n \geq 2$, $\pi_n\underset{\sim}{X}$ became the family of groups $\pi_n(X_n, X_{n-1}, z)$, $z \in X_0$. This led to the following definition [B-H 4].

A *crossed complex* C is a sequence

$$\ldots \longrightarrow C_n \xrightarrow{\delta} C_{n-1} \longrightarrow \ldots \longrightarrow C_2 \longrightarrow C_1 \underset{\delta^1}{\overset{\delta^0}{\rightrightarrows}} C_0$$

satisfying the following axioms.

(2.1) C_1 is a groupoid with C_0 as its set of vertices and δ^0, δ^1 as its initial and final maps.

We write $C_1(p, q)$ for the set of arrows from p to $q (p, q \in C_0)$ and $C_1(p)$ for the group $C_1(p, p)$.

(2.2) For $n \geq 2$, C_n is a family of groups $\{C_n(p)\}$ for $p \in C_0$ and for $n \geq 3$ the groups $C_n(p)$ are abelian.

(2.3) The groupoid C_1 operates on the right of each $C_n (n \geq 2)$ by an action denoted $(x, a) \mapsto x^a$. Here if $x \in C_n(p)$ and $a \in C_1(p, q)$ then $x^a \in C_n(q)$.

We use additive notation for all groups $C_n(p)$ and for the groupoid C_1 , and we use the same sumbol 0 for all their identity elements.

(2.4) For $n \geq 2$, $\delta : C_n \to C_{n-1}$ is a morphism of groupoids over C_0 and preserves the action of C_1 , where C_1 acts on the groups $C_1(p)$ by conjugation : $x^a = -a + x + a$.

(2.5) $\delta\delta = 0 : C_n \to C_{n-2}$ for $n \geq 3$ (and $\delta^0\delta^1 = \delta^1\delta : C_2 \to C_0$) .

(2.6) If $c \in C_2$, then δc acts trivially on C_n for $n \geq 3$ and operates on C_2 as conjugation by c , that is $x^{\delta c} = -c + x + c$ $(x, c \in C_2(p))$.

In the case when C_0 is a point we call C a *reduced* crossed complex. (Non-reduced crossed complexes were not considered till 1977.)

The question in 1975 was whether (reduced) crossed complexes were appropriate for a generalised Seifert-van Kampen theorem. The formulation and proof of such a theorem required a generalisation $\rho\underset{\sim}{X}$ of the double groupoid $\rho(X, Y, z)$ It was not clear that the structure possessed by $\rho\underset{\sim}{X}$ would be such as to give a category equivalent to that of crossed complexes.

<u>Remark</u>

The use of (reduced) crossed complexes for interpreting all the cohomology groups $H^n(G; A)$ of a group G with coefficients in a G-module A was found by various writers, notably Huebschmann, in his 1977 dissertation (cf. [Hu]), Holt [Ho] and Hill [Hiℓ](see the historical note by MacLane [MacL]). A different approach to this result is given by Loday in [Lo], together with a related interpretation of *hyper-relative groups*. The relation with non-abelian cohomology is described in [B-H 8].

Related results in the context of varieties of algebras (thus generalising the theory for groups) were shown by Lue (cf. [L]) but the relation with the classical theory was not made explicit. His work has recently been extended to *categories of interest* (in the sense of Orzech [O]) by Cegarra and Grandjean [C-G 1,2] and Garcia [G].

Thus the notion of crossed complex is seen to have wide significance. It would be interesting to see its use in other areas of mathematics, for

example, as suggested by Brown and Porter (unpublished), in the theory of deformations of structures, and, generally, for notions of higher homotopies (i.e. homotopies of homotopies, etc.).

An important step forward would be to find a use of crossed modules in a differential geometry setting.

3. T-complexes

The problem I suggested to Keith Dakin in 1975 (and so dragging him away from interesting work on topological groupoids and fibre bundles [D1]) was to find a simplicial version of double groupoids, and so in the process generalise double groupoids to all dimensions. Such a simplicial version seemed necessary because of

(i) the importance of simplicial methods in algebraic topology,

and (ii) their well-established literature.

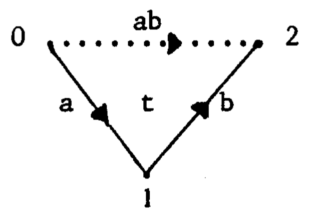
By contrast, as said above, many of the corresponding cubical techniques were unavailable, and apparently unknown.

It must be confessed that there were not many clues available. Two relevant ideas seemed to be

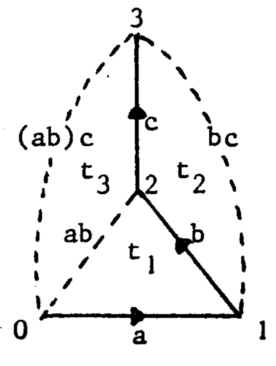
(i) the nerve NG of a groupoid G, as defined by Segal [S],

(ii) the idea of Kan complexes with canonical fillers.

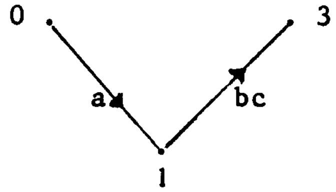
One easy point was that if a simplicial set K has canonical fillers of horns, then K_1 admits a partial multiplication. The product ab is defined when $\partial_0 a = \partial_1 b$, and is given by $ab = \partial_1 t$, where t fills the horn given by the solid lines in the following diagram.



Dakin noticed that to obtain associativity of this multiplication, a further condition on the canonical fillers is necessary, as shown by the diagram



in which t_1 , t_2 , t_3 are canonical fillers, given in that order. If t fills this horn, then to obtain $(ab)c = a(bc)$ we must require that the face $\partial_2 t$ of t is the canonical filler of the horn, shown as:



Dakin proposed a generalisation of this idea to all dimensions, as follows.

Let K be a simplicial set, and for $n \geq 0$ let H^n be the set of all horns in K ; that is, H^n is the set of all maps $\Lambda_k^n \to K$ for all $0 \leq k \leq n$. Note that if $x \in K_n$, then the collection of faces of x except the k'th defines a horn $\Lambda_k^n \to K$, which we call the k-*horn* of x . Suppose given the additional structure of a function $F^n : H^n \to K_n$ assigning to each horn $h : \Lambda_k^n \to K$ a *canonical filler* $F^n(h) \in K_n$. The axioms shall be the following:

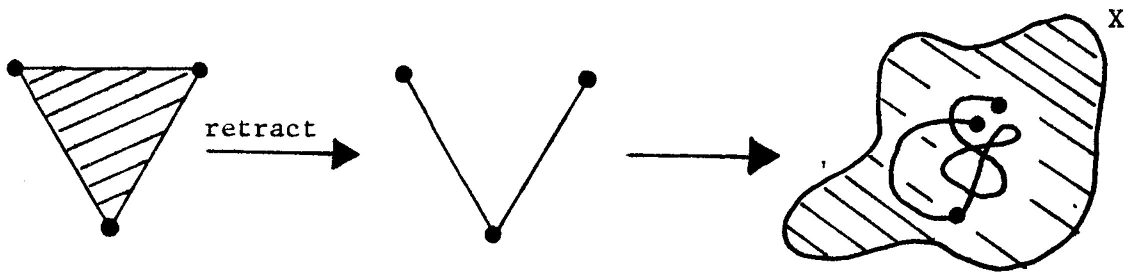
CF1) *If* x *is the canonical filler of one of its horns, then it is the canonical filler of any of its horns.*

CF2) *A degenerate element is a canonical filler of any of its horns.*

CF3) *If* x *is a canonical filler, and all its faces* $\partial_i x$ *, except possibly* $\partial_k x$ *, are canonical fillers, then* $\partial_k x$ *also is a canonical filler.*

Dakin proved that these axioms imply that the induced multiplication on K_1 is a groupoid structure.

He lectured on this material at Bangor in September, 1975. Philip Higgins, who was then visiting Bangor, proposed a reformulation of the axioms in terms of the subsets of K_n consisting of the canonical fillers. Later, we agreed to call these fillers *thin elements*, the motivation being that canonical fillers in the singular complex SX come from a retraction



and so such fillers, while not necessarily degenerate, are certainly not 'thick'. The axioms for this structure then became as follows.

<u>Definition</u> (K. Dakin)

A T-*complex* consists of a simplicial set K together with subsets $T_n \subset K_n$, $n \geq 1$, of elements called *thin*, and satisfying the following axioms

(T1) *All degenerate elements are thin.*
(T2) *Any horn in* K *has a unique thin filler.*
(T3) *If all faces but one of a thin element are thin, so also is the remaining face.*

These marvellous axioms suggested immediately that T-complexes ought to be an appropriate simplicial model of double groupoids. Further, the axioms are easily given in a cubical context, and such cubical T-complexes seemed a possible generalisation to all dimensions of double groupoids. In any case, it was clearly essential to analyse the consequences of the axioms.

Remark 1.

The first form of Dakin's axioms might be more appropriate for setting up theories of T-complexes in other categories than that of sets. For such a category, one would need to have the *object* H^n_k of k-horns of K_n defined, and then require that $F^n_k : H^n_k \to K_n$ (which gives the canonical fillers) should be a morphism in the category. More work is needed in this direction.

Remark 2.

There are related ideas in Duskin's Memoir [Du] and in the subsequent thesis of his student Paul Glenn [Gℓ]. Paragraph two on p.4 of [Du] looks to be a precursor of the crossed complex interpretation of group or Lie algebra cohomology (cf. the Remark in §2 above) and also a step towards the equivalence of crossed complexes and T-complexes. The notion of canonical fillers in a simplicial objects in a category with finite limits is discussed in [Gℓ] in relation to the notion of *hypergroupoid*, and in talks by Duskin at various category theory conferences. In a talk at Gummersbach (July, 1981) (cf. [K-P-T]), Duskin explained how simplicial T-complexes were equivalent to filtered systems of hypergroupoids; he also gave his own account of the equivalence between crossed complexes and simplicial T-complexes, using an intermediate category of simplicial groups in which each horn has a unique filler which is a product of degenerate elements. (See also [Du2].)

4. K. Dakin's thesis

The immediate problems posed by the notion of a T-complex were two.

Problem 1

Give a good geometric example of a T-complex, analogous to the double groupoid $\rho(X, Y, z)$ of a triple.

Problem 2

Find a more convenient algebraic presentation of T-complexes, analogous

to the equivalence between double groupoids and crossed modules.

By this time, the expectations for geometric examples had changed crucially from those suggested by Problems (1.2)-(1.3) of §1. It was now clear that one should look for relative rather than absolute situations. This in itself was curious, since initially the good thing about the fundamental groupoid seemed to be that it enabled one to get rid of base points. Only gradually did it emerge that it was useful, not to get rid of base points, but to allow more flexibility by choosing a set X_0 of base points, and so defining a fundamental groupoid $\pi_1(X, X_0)$.

Thus in the present case, when structure in all dimensions is considered, it was natural to generalise the 2-dimensional situation by considering a filtered space

$$\underset{\approx}{X} : X_0 \subset X_1 \subset \ldots \subset X_n \subset \ldots \subset X$$

and the *relative* singular complex $R^\Delta \underset{\approx}{X} \subset SX$ of all filtered maps $f : \Delta^n \to \underset{\approx}{X}$, where $\underset{\approx}{\Delta}^n$ has its filtration by r-skeletons $\Delta^{n,r}$, so that f is required to satisfy $f(\Delta^{n,r}) \subset X_r$, $r \geq 0$. Then $R^\Delta_n \underset{\approx}{X}$ may be given the compact-open topology and one can set $\rho^\Delta_n \underset{\approx}{X} = \pi_0 R^\Delta_n \underset{\approx}{X}$. (In order to distinguish the simplicial functor ρ defined in [As] from the cubical functor ρ defined in [B-H 5] we write these respectively as ρ^Δ and $\rho^\Box$.) Clearly $\rho^\Delta \underset{\approx}{X}$ inherits the structure of simplicial set from SX and $R^\Delta \underset{\approx}{X}$. Define an element of $\rho^\Delta_n \underset{\approx}{X}$ to be *thin* if it has a representative $f : \underset{\approx}{\Delta}^n \to \underset{\approx}{X}$ such that $f(\Delta^n) \subset X_{n-1}$. Is then $\rho^\Delta \underset{\approx}{X}$, with this structure, a T-complex, assuming that each loop in X_0 is contractible in X_1 ?

This problem proved very hard. In the cubical case, it was solved affirmatively in the summer of 1976 by Brown and Higgins, but required a new set of techniques, involving the use of collapsings of subcomplexes of I^n.

Dakin's main work is on the algebraic problem. The candidate available for an algebraic structure equivalent to that of T-complex was that of crossed complexes, and it was in this direction his results tend.

Define a T-complex K to be of rank $\leq n$ if all elements of K_m are thin for $m > n$. Let $K^{(n)}$ denote the smallest sub-T-complex of K containing $K_0, K_1, \ldots, K_n$. Then the $K^{(n)}$ are of rank $\leq n$ and define a filtration $\underset{\approx}{K}$ of K. Since each $K^{(n)}$ is a T-complex, it is also a Kan complex.

Dakin, in Ch.4 of [D2], refines standard simplicial work to show that, as in the topological case, the fundamental groupoid $\pi_1(K^{(1)}, K^{(0)})$ and relative homotopy groups $\pi_n(K^{(n)}, K^{(n-1)}, p)$, $n \geq 2$, $p \in K_0$, fit together

to form the homotopy crossed complex $\pi\underset{\sim}{K}$. He also proves that $\pi_i(K^{(n)}, p) = 0$ for $i > n$ (cf. p.47). (This fact uses only Axioms T1 and T2.)

This filtration is thus quite remarkable, and Loday has suggested for it the name "*inverse* Postnikov system". The point here is that the usual construction from a Kan complex X of a complex $X^{(n)}$ with $\pi_i(X^{(n)}) = 0$ for $i > n$ is that of a *fibration* $X \to X^{(n)}$, and not of a *subcomplex* of X.

In Ch.2 of [D2], Dakin shows that the nerve of a groupoid gives an equivalence between the category of groupoids and the category of T-complexes of rank 1.

In Ch.3 of [D2], he explores the algebraicstructures in dimensions >1 implied by the T-complex axioms. He shows that K_n admits a groupoid structure $\underset{i}{+}$ with initial, final maps and identity maps ∂_i, ∂_{i-1} and s_{i-1} respectively for $1 \leq i \leq n$.

This area is one which clearly requires more work. For example the interchange law is not discussed. Ideally, one would like a list of such algebraic structures, with possibly some extra structure, which would imply the T-complex axioms. This would be analogous to the cubical case, in which there is an isomorphism of categories between cubical T-complexes and ω-groupoids [B-H 3,5].

However, Ch.3 does show important features of T-complexes. A philosophical point is that in T-complexes both algebraic operations and rules on algebraic operations are determined by particular simplices. Thus for the operation $\underset{i}{-}$ in dimension n, there is for each $a \in K_n$ a simplex $Ia \in K_{n+1}$ one face of which is a, another face is $\underset{i}{-}a$, and another face is $s_i\partial_i a$, in such a way that Ia gives the rule $a \underset{i}{-} a = s_i\partial_i a$. This idea is of course implicit in traditional work on Kan complexes; it is just that T-complexes have a more algebraic character.

Ch.4 of [D2] has been discussed above.

In Ch.5 of [D2], Dakin shows that the functor $K \mapsto \pi\underset{\sim}{K}$ from T-complexes to crossed complexes gives an equivalence between T-complexes of rank ≤ 2 and crossed modules. This shows that T-complexes do give a simplicial version of double groupoids. His methods, though, are diagrammatic and do not directly generalise to higher dimensions.

5. Progress on ω-groupoids

By October 1976, Brown and Higgins had made progress in the cubical version say $\rho^{\square}\underset{\sim}{X}$, of the above homotopy functor on filtered spaces, and had shown that

this has the structure of cubical T-complex if $\underset{\approx}{X}$ is a J_0-filtered space (i.e. if each loop in X_0 is contractible in X_1). The proofs of all the steps in the colimit theorem for $\rho^{\square}\underset{\approx}{X}$ and for $\pi\underset{\approx}{X}$ were completed by July, 1977, and the results were announced in [B-H 2,3]. The main steps in the proof were (cf. [B-H 4,5]):

(5.1) The *definition* of a category of ω-*groupoids*, and the *proof* that $\rho^{\square}\underset{\approx}{X}$ is an example of such an ω-groupoid.

(5.2) The *proof* of the equivalence of categories of ω-groupoids and crossed complexes.
Here if G is an ω-groupoid, then it contains a crossed complex γG and for $n \geq 2$ there is a *folding operation* $\Phi : G_n \to (\gamma G)_n$ with convenient algebraic properties. Also G can be recovered from γG (by "unfolding" inductively). Further G has the structure of cubical T-complex, the "thin" elements of G being precisely those which fold to zero in γG.

This T-complex structure, which is established by means of a purely algebraic version of the cubical homotopy addition lemma, is a key step in the proof of the colimit theorem. The reason is as follows. It is required to prove that the construction of an element $F\alpha$ of an ω-groupoid G from a map $\alpha : \underset{\approx}{I}^n \to \underset{\approx}{X}$ is independent of the choices made. Different choices lead to elements $F\theta$, $F\theta'$, but θ and θ' are related by a homotopy $H : \theta \equiv \theta'$ which may itself be deformed in a nice way to a 'good' homotopy $K : \phi \equiv \phi'$ where $F\theta = F\phi$, $F\theta' = F\phi'$. The homotopy K is 'good' because it determines an element x of G_{n+1} with $\partial^0_{n+1}x = F\phi$, $\partial^1_{n+1}x = F\phi'$. The proof is completed by showing x is degenerate of the form $x = \varepsilon_{n+1}\partial^0_{n+1}x$. This uses a characterisation of degenerate elements in terms of thin elements [B-H 5; Proposition 4.4] whose proof uses the axioms T1 and T2 for the above thin elements. That x is degenerate implies $F\phi = F\phi'$.

(5.3) The *fibration theorem*. This says that if $\underset{\approx}{X}$ is a J_0-filtered space, and $R^{\square}\underset{\approx}{X}$ is the cubical relative singular complex of $\underset{\approx}{X}$ (which in dimension n consists of all filtered maps $\underset{\approx}{I}^n \to \underset{\approx}{X}$), then the projection $R^{\square}\underset{\approx}{X} \to \rho^{\square}\underset{\approx}{X}$ is a Kan fibration of cubical complexes. This theorem allows for a crucial characterisation of thin elements in $\rho^{\square}\underset{\approx}{X}$, and helps to establish the connection between $\rho^{\square}\underset{\approx}{X}$ and $\pi\underset{\approx}{X}$.

Given these results, the *proof* of the colimit theorem for $\rho^{\square}\underset{\approx}{X}$ (and hence for $\pi\underset{\approx}{X}$) is a direct generalisation of the proof of the 1-dimensional Seifert-van Kampen theorem (see [B5] for a sketch of the ideas).

In the next section we show how Ashley established simplicial analogues of (5.1), (5.2) and (5.3). However we note that a direct proof of the colimit

theorem for $\rho^{\Delta}\underset{\sim}{X}$ is still not available, as simplicial methods lack the multiple compositions of cubical theory.

6. Ashley's Thesis

Nicholas Ashley came to Bangor in October, 1976, as S.R.C. Research Student, having previously completed an M.Sc. in algebraic topology at Manchester, with a dissertation on differential structures on products of spheres. I gave him the problem of characterising simplicial T-complexes, and he set about it in his own quiet and completely independent way. He lectured on his solution at Bangor in October, 1977.

Chapter 1 of [As], of 45 pages, is devoted to proving the equivalence of the categories of simplicial T-complexes and of crossed complexes. Let us denote these categories by ST and XC respectively.

His functor $D : \mathsf{XC} \to \mathsf{ST}$ which is given in Ch.1, §11, generalises a functor K to simplicial sets given by Blakers [Bℓ] for the case of reduced crossed complexes. (The only other reference I have found on this functor is [An].)

In the next section, I will give an alternative formulation of this functor, and a proof, using the colimit theorem for homotopy crossed complexes, that its values are T-complexes.

Ashley's construction of a functor $N : \mathsf{ST} \to \mathsf{XC}$ is quite complex, and in particular he eschews Dakin's methods, based on Lamotke's exposition of simplicial theory [La]. Instead he makes a refined analysis of some of the rich algebraic structure of T-complexes.

In 1.1 he reworks part of Dakin's results on the groupoid structures on a T-complex K, and gives the groupoid structure $\underset{n}{+}$ on K_n, which he writes as $(x, y) \mapsto xy$. This groupoid structure has a family K^n_v of vertex groups for all $v \in K_{n-1}$. In Ch.1 §2 he constructs an isomorphism $f[v] : K^n_v \to K^n_{v'}$, where $v' = s_{n-2}d_0v$ and this gives an isomorphism $\hat{f}[v] : K^n_v \to K^n_{\beta v}$ where βv is the vertex $d_0^{n-1}v$ of v. From this is obtained an isomorphism $h[u, v] : K^n_u \to K^n_v$ for any $u, v \in K_{n-1}$ with $\beta u = \beta v$ (Theorem 2.1).

Sections 3-7 construct the crossed complex $N(K, T)$ of a T-complex K with thin elements T. If we call this crossed complex C, then $C_0 = K_0$, $C_1 = K_1$ with the groupoid structure mentioned above. If $a \in K_0$, and $n \geq 1$, then K^n_a denotes the group K^n_v where $v = s_0^{n-1}a$. Then C_2 is the family of groups K^2_a, $a \in K_0$. For $n \geq 3$, C_n is the family of subgroups $K^n_a(A)$ of K^n_a consisting of those $x \in K^n_a$ such that $d_ix = s_0^{n-1}a$ for $i > 0$. For technical reasons he also considers groups $K^n_a(B)$ for $n \geq 2$, defined

inductively by $K_a^2(B) = K_a^2$, $K_a^n(B) = \{x \in K_a^n : d_i x \in K_a^{n-1}(B)$ for $0 \le i \le n-2\}$. His main results in §3 are that $K_a^n(A)$, $K_a^n(B)$ are both in the centre of K_a^n for $n \ge 2$, and K_a^n is the product group $K_a^n(A) \times (T_n \cap K_a^n)$.

§4 defines a boundary $\delta : K_a^n \to K_a^{n-1}$. §5 constructs a groupoid action ϕ of K^1 on the family K_a^n , $n \ge 2$. §6 relates the isomorphisms ϕ and h . These results are used in §7 to prove $C = N(K, T)$ is a crossed complex.

§8 uses the groupoid structure to define a function $k_n : K_n \to K_n$ with properties set out in 8.3. (For a T-complex J in §9, the corresponding function is written j_n.) This function is used in §9 to show that for a T-complexes J, K , any morphism $f : N(J) \to N(K)$ of crossed complexes extends uniquely to a morphism $J \to K$ of T-complexes. §10 proves the "homotopy addition lemma" in T-complexes, using crucially a function $\mu_n : K_n \to K_n$ which corresponds to the folding map Φ of ω-groupoids.

§11 defines a functor from crossed complexes to T-complexes which is an inverse equivalence to the functor N described above. This functor is written D by Ashley, and is in fact the same for reduced crossed complexes as a functor K , from reduced crossed complexes to simplicial sets, described by Blakers in [Bℓ]. Of course Blakers does not have the notion of T-complex. The definition of thin elements in DC is easy, but the verification of the axioms is a tricky technical exercise. (A different proof is given in our next section.)

The proof that D is an inverse equivalence of N is given in §12 (using much of the material of previous sections).

Chapter 2 gives a simplicial and generalised version of the Brown-Higgins functor $\rho^{\square}\underset{\sim}{X}$ of a J_0-filtered space $\underset{\sim}{X}$. Ashley considers increasing sequences

$$\underset{\sim}{K} : K^0 \subset K^1 \subset \dots \subset K^n \subset \dots$$

of Kan complexes such that $\underset{\sim}{K}$ is J_0 (i.e. each simplicial map $\dot{\Delta}(2) \to K^0$ extends to a simplicial map $\Delta(2) \to K^1$) and such that for $n \ge 1$

$$d_i K_n^j \subset K_{n-1}^{j-1} \quad \text{for} \quad 0 \le i \le n \le j \quad .$$

He then sets $(R\underset{\sim}{K})_n = K_n^n$, $n \ge 0$, and finds $R\underset{\sim}{K}$ inherits the structure of Kan complex. He also sets $T^1\underset{\sim}{K}$ to be K_n^{n-1} in dimension; then $T^1\underset{\sim}{K}$ is a graded subset of $R\underset{\sim}{K}$. He gives a notion $\equiv$ of *filter homotopy* between elements of $(R\underset{\sim}{K})_n$, and finds that the quotient $\rho\underset{\sim}{K} = (R\underset{\sim}{K})/\equiv$ is a simplicial set. The image of $T^1\underset{\sim}{K}$ in $\rho\underset{\sim}{K}$ is written $T\underset{\sim}{K}$, and the elements of $T\underset{\sim}{K}$ are called the *thin* elements of $\rho\underset{\sim}{K}$. As in the cubical case, a key result is that the

projection $R\underset{\approx}{K} \to \rho\underset{\approx}{K}$ is a Kan fibration (Theorem 2.10) and this allows for a proof that $(\rho\underset{\approx}{K}, T\underset{\approx}{K})$ is a T-complex (Theorem 2.13).

Chapter 3 relates T-complex theory to that of simplicial groups. If G is a simplicial group, a product of degenerate elements in G is called *slim*. The slim elements form a graded subset D of G. Any horn in G has a slim filler (this is Moore's result referred to earlier). In general, though, (G, D) is not a T-complex (1.3, 1.4 and 1.9), although it is if G is abelian (1.5). §2 of Chapter 3 defines, using simplicial groups, a category equivalent to that of T-complexes.

Chapter 4 contains a miscellany of small results.

After discussions with Duskin in 1979, Ashley has found another account of the main result of Ch.1 for reduced crossed complexes. If G is a simplicial group, let $M'G$ denote its Moore chain complex [Ma, p.68], reindexed so that M'_nG is the intersection of $\mathrm{Ker}(\partial_i : G_{n+1} \to G_n)$ for $i > 0$, and setting $M'_0G = 0$. Then Ashley shows $M'G$ has the structure of crossed complex (with some canonical actions) if and only if in G *any horn has a unique slim filler*. He also shows that G has this last property if and only if $\bar{W}G$, with a canonical thin structure, is a T-complex.

7. <u>The classifying space of a crossed complex</u>

In this section we denote the category of crossed complexes by XC, of simplicial sets by S and of simplicial T-complexes by ST. Also, to conform with other usage, we write M for the functor $ST \to XC$ (as this is thought of as a kind of Moore chain complex) and the functor $XC \to ST$ is written N and called the *nerve* (as it is a direct generalisation of the nerve of a groupoid, as defined in [Se]).

The two theses by Dakin and Ashley accept the problem of simplicial T-complexes on its own terms.

The idea of T-complex seems interesting, and the algebra that these deceptively simple axioms imply has considerable fascination. Ashley's result (that Blakers' functor from crossed complexes to simplicial sets in fact determines an equivalence N between the categories of crossed complexes and of simplicial T-complexes) is an impressive technical achievement. However, the definition of this functor N seems very technical, and no application is given of the fact that if C is a crossed complex, then $N(C)$ is not just a simplicial set, but also has the structure of T-complex.

The object of this section is to give some idea of known and of possible applications of these results.

One point should be dealt with first. Ashley claims (p.(ii)) that the equivalence between ST and XC generalises the theorem of Dold-Kan giving an equivalence between the categories of simplicial abelian groups and of chain complexes. However, this fact is never made explicit, and we first explain this, describing results of [B-H 9].

Let C denote the category of chain complexes. If L is a chain complex, then a crossed complex $B = i(L)$ is defined as follows. In dimension 0, B_0 is the set L_0; in dimension 1, the groupoid B_1 has set of arrows $L_0 \times L_1$ with initial and final maps, respectively $(x, a) \mapsto x$, $(x, a) \mapsto x + \partial a$, and groupoid structure

$$(x, a) + (x + \partial a, b) = (x, a + b) ;$$

in dimensions $n \geq 2$, B_n is the family $L_0 \times L_n$ of abelian groups $\{x\} \times L_n$ indexed by $x \in L_0$. The operation of B_1 on B_n is given by

$$(x, b)^{(x, a)} = (x + \partial a, b) \quad x \in L_0 , a \in L_1 , b \in L_n .$$

The boundary $\delta : B_n \to B_{n-1}$ $(n \geq 2)$ is induced by $\partial : L_n \to L_{n-1}$. The axioms for a crossed complex are easily verified. Clearly $L \mapsto i(L)$ extends to a functor $i : C \to XC$. (Compare §6 of [Gr].)

(7.1) *Theorem* [B-H 9] *The functor* $i : C \to XC$ *has a left adjoint* r. *Also, if* X *is a simplicial set, and* $|\underline{X}|$ *is the geometric realisation of* X *with its filtration by skeletons, then* $r\pi|\underline{X}|$ *is naturally isomorphic to* $C_N(X)$, *the chain complex of normalised chains of* X.

Let S denote the category of simplicial set. Then there is a forgetful functor $U : ST \to S$. We need the following result.

(7.2) *Theorem* *The forgetful functor* $U : ST \to S$ *has left adjoint the functor*

$$\begin{aligned} S &\longrightarrow ST \\ X &\longmapsto \rho^{\Delta}|\underline{X}| . \end{aligned}$$

This result says that $\rho^{\Delta}|\underline{X}|$ is the *free* T-*complex on the simplicial set* X.

There is a corresponding cubical result which is *not* stated in [B-H 5] - all that is stated (in §9) is that $\rho^{\square}(\underline{I}^n)$ is the free ω-groupoid on one generator c^n of dimension n. However, the methods given there are easily extended to prove the more general result, and all these methods carry over easily to the simplicial case. In a similar manner to (9.5) of [B-H 5] we obtain:

(7.3) *Corollary: The functor* $N : XC \to ST$ *is equivalent to the functor* N' *given by*

$$N'_n C = XC(\pi|\underline{\Delta}^n|, C) ,$$

with simplicial operators given by maps on Δ^n .

(7.4) Remark. It is easy to see from the homotopy addition lemma that the functor N' defined in (7.3) is a reformulation of Blakers functor K in [Bℓ], and used by Ashley [As]. Define a morphism $f : \pi|\underline{\Delta}^n| \to C$ to be *thin* if $f(\delta^n) = 0$ where δ^n is a generator of $\pi_n(|\underline{\Delta}^n|)$. It is interesting to give a proof that this thin structure makes N'(C) a T-complex by using a colimit theorem ((5.2) Corollary) from [B-H 5]. The key point is to prove N'(C) a Kan complex. Let (x_i) be an (n, k)-horn in N'(C) . Let δ_i^n denote the subcomplex of Δ^n generated by $\partial_i\delta^n$. The (x_i) can be taken as maps $\pi|\underline{\partial_i\delta^n}| \to C$ which agree on the images of $\pi|\underline{\partial_i\partial_j\delta^n}|$ $(i < j,\ i, j \neq k)$. By Corollary (5.2) of [B-H 5], the coequaliser of the pair of morphisms of crossed complexes

$$\bigsqcup_{\substack{i<j \\ i,j\neq k}} \pi|\underline{\partial_i\partial_i\delta^n}| \rightrightarrows \bigsqcup_{i\neq k} |\underline{\partial_i\delta^n}|$$

is just $\pi|\underline{\Lambda}_k^n|$. Hence, giving an (n, k)-horn in N'(C) is equivalent to giving a morphism $\pi|\underline{\Lambda}_k^n| \to C$. Since $|\Lambda_k^n|$ is a retract of $|\Delta^n|$, and any such retraction kills $\pi_n|\underline{\Delta}^n|$, we find any horn in N'C has a thin filler f . This filler is unique because the value of f on the element of $\pi_{n-1}|\underline{\Delta}^n|$ determined by $\partial_k\delta^n$ is given by the homotopy addition lemma (since f kills $\pi_n|\underline{\Delta}^n|$). Finally, Axiom T3 again follows from the homotopy addition lemma.

(7.5) *Corollary: If* L *is a chain complex, then* Ni(L) *is naturally isomorphic to the simplicial abelian group given in dimension* n *by*

$$C(C_N(\Delta^n),\ L)\ .$$

Proof.

$$\begin{aligned} Ni(L)_n &\cong XC(\pi|\underline{\Delta}^n|,\ iL) \\ &\cong C(r\pi|\underline{\Delta}^n|,\ L) \quad \text{by adjointness in (7.1)} \\ &\cong C(C_n(\Delta^n),\ L) \quad \text{by (7.1)}. \quad \square \end{aligned}$$

The functor from chain complexes to simplicial abelian groups given in (7.4) is called the Dold-Kan functor [Do, K6]. We write it K . It gives an equivalence between these categories.

The purpose of giving this last result is to show how (7.2) enables methods in simplicial abelian groups to be generalised, by the use of crossed complexes, to non-abelian twisted cases.

Recall first of all that the Dold-Kan functor K generalises the construction of Eilenberg-MacLane spaces, since if A is an abelian group, and s^nA is the chain complex which is A in dimension n and zero elsewhere, then $K(s^nA)$ is isomorphic to Eilenberg-MacLane's construction of K(A, n) .

Let SA denote the category of simplicial abelian groups. The forgetful functor $U : SA \to S$ has a left-adjoint C, where $(CX)_n$ is the free abelian group on X_n. Note that the inverse equivalence of $K : C \to SA$ is the normalised chain complex M, and $MC(X) = C_N(X)$, the complex of normalised chains of X. So we have that if L is a chain complex.

$$\begin{aligned} S(X, UKL) &\cong SA(CX, KL) \\ &\cong C(C_N(X), L) . \end{aligned}$$

This bijection passes to homotopy classes:

$$[X, UKL] \cong [C_N(X), L] ,$$

and the latter abelian group is (cf. [B1]) $H^0(X; L)$, the *cohomology of* X *with coefficients in* L. In particular, if $L = s^n A$, where A is an abelian group, we obtain the classical bijection

$$[X, K(A, n)] \cong H^n(X; A) .$$

These methods are exploited in [B2] to give information on Postnikov invariants of function spaces.

Now let C be a crossed complex, and X a simplicial set. Then we have

$$\begin{aligned} S(X, UNC) &\cong ST(\rho^\Delta |\underline{X}|, NC) \\ &\cong XC(\pi|\underline{X}|, C) . \end{aligned}$$

In the cubical case, it has been checked that this bijection carries over to homotopy classes. Presumably, the same applies to the simplicial case. So we have

$$[X, UNC] \cong [\pi|\underline{X}|, C] . \tag{7.6}$$

The latter set can be interpreted as $H^0(X, C)$, the *cohomology of* X *with coefficients in the crossed complex* C.

A particular case of a crossed complex is $s^1 G$, where G is a group. (That is, $s^1 G$ is G in dimension 1 and zero elsewhere.) Then $N(s^1 G)$ is just the classical $K(G, 1)$, and (7.6) determines $[X, K(G, 1)]$ in terms of non-abelian cohomology. More generally, let G be a group and A a G-module. For $n \geq 2$, let $C = C(n, G, A)$ be the crossed complex with $C_1 = G$, $C_n = A$ (with the given G-action) and all other C_i and all boundaries zero. Then the set $H^0(X, C(n, G, A))$ can be interpreted as the *n'th cohomology of* X *for all local systems on* X *via* G, since the classification is over all morphisms $\pi_1|X| \to G$. Alternatively, we can fix a morphism $\theta : \pi_1|X| \to G$ and consider only morphisms $\pi|\underline{X}| \to C$ which induce θ; this gives us $H^n_\theta(X; A)$, the cohomology with coefficients in the local system A determined by θ. More generally, the group G may be replaced by a groupoid (for example the groupoid

$\pi_1|X|)$; we can consider crossed modules, instead of modules; or it may be convenient to consider the most general case, as in (7.6).

It is proved in [B-H 9] that $\pi\underline{Y}$ is a homotopy invariant of $\underline{Y}$ (for a suitable definition of homotopy). Let us write BC for the geometrical realisation $|NC|$ of NC, when C is a crossed complex, and call BC the *classifying space* of C. Let $\underline{Y}$ be the skeletal filtration of a CW-complex Y. We deduce from (7.5) and homotopy invariance that

$$[Y, BC] \cong [\pi\underline{Y}, C] \,, \tag{7.7}$$

and again we interpret the latter set as $H^0(Y; C)$.

As an example, suppose we wish to compute $H = \pi_1(K(G, 1)^Y, f)$ for a group G, reduced CW-complex Y and map $f : Y \to K(G, 1)$ (as has been done in [Go], [Ha]). We can equate H with $[Y \times I, K(G, 1); u]$, the group of homotopy classes rel u of maps $Y \times I \to K(G, 1)$, where u is the composite of the projection $Y \times \mathring{I} \to Y$ and $f : Y \to K(G, 1)$. By a relative version of (7.6), this set is bijective with

$$[\pi(\underline{Y} \times \underline{I}), s^1G; \pi u] \,.$$

Now $\pi_1(\underline{Y} \times \underline{I})$ is the free product of the free group $\pi_1(Y^1, y)$ (where $Y^0 = \{y\}$) and the groupoid $\pi_1(I, \mathring{I})$, which is free on one generator ι. The 2-cells of $Y \times I$ are $e^2_\lambda \times \{0\}$, $e^2_\lambda \times \{1\}$ and $e^1_\mu \times \mathring{I}$ for all 2-cells e^2_λ and 1-cells e^1_μ of Y.

$$\begin{tikzcd}
(y, 1) \arrow[r, "e^1_\mu \times 1"] & (y, 1) \\
(y, 0) \arrow[r, "e^1_\lambda \times 0"'] \arrow[u, "y \times I"] & (y, 0) \arrow[u, "y \times I"']
\end{tikzcd}$$

Let $\alpha : \pi(\underline{Y} \times \underline{I}) \to s^1G$ be a map of crossed complexes extending $\pi u : \pi(\underline{Y} \times \mathring{I}) \to S^1G$, and let $\alpha(\iota) = z \in G$. The commutativity of the diagram

$$\begin{tikzcd}
\pi_2(\underline{Y} \times \underline{I}) \arrow[r, "\alpha_2"] \arrow[d, "\delta"'] & 0 \arrow[d] \\
\pi_1(\underline{Y} \times \underline{I}) \arrow[r, "\alpha_1"] & G
\end{tikzcd}$$

and the fact that α extends πu, implies $zf_*(e^1_\mu) = f_*(e^1_\mu)z$ for all μ, and hence z belongs to the centraliser of $f_*(\pi_1(Y, y))$ in G. Conversely, any such z determines an α extending πu. All homotopies rel πu are

in this case constant, and we deduce Gottlieb's result that $\pi_1(K(G, 1)^Y, f)$ is isomorphic to the centraliser of $f_*(\pi_1(Y, y))$ in G .

One can go further than (7.7) to a more computational result. According to [B-H 9] there is a functor $\Delta : XC \to GC$, where the latter category is that of chain complexes with groupoids as operators. If $\underline{Y}$ is the skeletal filtration of a reduced CW-complex Y , then $\Delta\pi\underline{Y}$ is the chain complex $C_*(\tilde{Y})$ of cellular chains of the universal cover $\tilde{Y}$ of Y , with the fundamental group of Y as group of operators on $C_*(\tilde{Y})$. In [B-H 9] we extend results of [W] to obtain a bijection

$$[\pi\underline{Y}, C] \cong [C_*(\tilde{Y}), \Delta C]$$

for a suitable definition of the latter set of homotopy classes. Putting this together with (7.6) and using the homotopy invariance of $\pi\underline{Y}$ (so that we do not need to assume $Y = |X|$ for a simplicial set X) we obtain a bijection

$$[Y, BC] \cong [C_*(\tilde{Y}), \Delta C] . \qquad (7.8)$$

The proof of the bijection (7.8), and its applications, illustrate a remark of Whitehead in the Introduction to [W], which reads in our terminology:

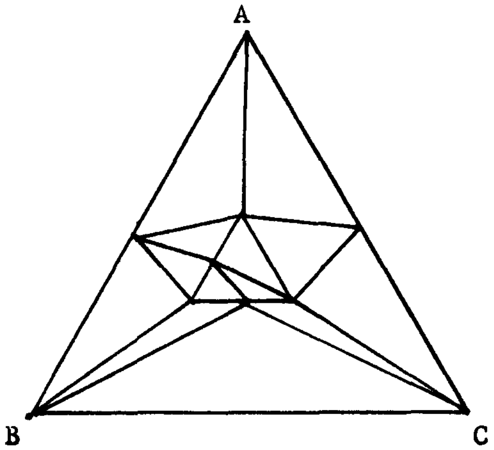
The crossed complex $\pi\underline{Y}$ *appears to be more useful than the chain complex* $C_*(\tilde{Y})$ *in problems concerning geometrical realisability. On the other hand, the chain complex* $C_*(\tilde{Y})$ *is convenient in studying concrete problems.*

One of Whitehead's illustrations of the last sentence is his discussion in §15 of [W] of the homotopy type of lens spaces. An application to maps of surfaces is given in [B6].

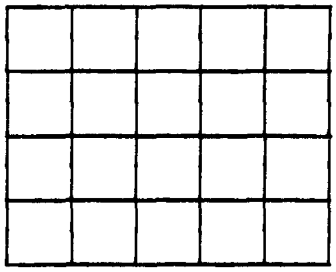
It is also interesting that the proof of (7.8) uses the category of simplicial T-complexes, which is itself one of a family of categories equivalent to crossed complexes. The study of these categories has a basic motivation:

Determine an algebraic operation inverse to subdivision.

Subdivision is of course an old technique in topology. The idea is to study a space by cutting it up into small, manageable bits. Intuitively, one then studies *cycles* and *boundaries* as certain "composites" of these bits. Eventually, it was found convenient to treat these "composites" as formal sums, and this is the formulation of homology theory that we know today. It is, indeed, difficult to know how one can "really compose" all the bits of the following subdivision of the triangle ABC in order to form the big triangle ABC. Simplicial theory lacks suitable composition operations.



By contrast, in cubical theory, such compositions are easy to manage, since in a diagram such as



one composes rows first and then columns, and this is a well-defined operation. The interchange law allows one to carry out these operations in the other order, or by computing blocks in a partitioned matrix.

A theory of general compositions, including simplicial, cubical, or polyhedral "pieces" or "bits", has to do three things:

<u>Composition 1</u>. Define the "bits" and the circumstances under which they are "composable".

<u>Composition 2</u>. Given "composable bits", define their "composite".

<u>Composition 3</u>. Give all relations among various "compositions".

It seems likely that all three of these requirements are met by the theory of poly-T-complexes [J]. Thus the notion of T-complex looks as if it will continue to have wide ramifications.

8. <u>Problems</u>

(8.1) Using the notion of simplicial object with canonical fillers, one can define a T-complex object in any category with finite limits, so that an object of horns is well defined, as in [Gℓ]. Is the equivalence between T-complexes and crossed complexes valid in any category with finite limits, or does one need the categories of interest of [O] , [C-G 1,2], [Ga]?

(8.2) It is proved in [B-H 9] that the equivalence between ω-groupoids and crossed complexes can be extended to an equivalence between categories with homotopy (i.e. between 2-categories). At present, the corresponding notions of homotopy have not been worked out for simplicial T-complexes.

(8.3) It is known that crossed complexes form a cartesian closed category [How]. It follows that simplicial T-complexes form a cartesian closed category. It would be interesting to write down explicitly this structure on ST .

(8.4) The cartesian closed structure on XC mentioned in (8.3) does not give the internal hom functor one would like. The right structure should be a monoidal closed structure with $\otimes$ and hom in which $\hom(C, D)_n$ is the homotopies of level n of morphisms $C \to D$.

(8.5) It is known from [As] that simplicial abelian groups are special cases of T-complexes. The corresponding cubical notions have not been worked out. Is the category of chain complexes equivalent to the abelian group objects in the category of cubical sets with connection (for which, see [B-H 4])?

(8.6) Some of the techniques of ω-groupoids and cubical T-complexes are tantalisingly reminiscent of the theory of infinite loop spaces, particularly May's "little cube operad" [Ma 2], [Ad]. Is there some synthesis of the two theories? A start might be to *replace the pointed spaces used in infinite loop space theory, by filtered spaces.*

(8.7) Can crossed complexes be used in discussing duality in manifolds? An obstacle here is that cup or cap products in the usual homology and cohomology require a diagonal map at the chain level; for crossed complexes, an appropriate tensor product is at present lacking.

(8.8) In view of the utility of crossed complexes in the cohomology of groups and in homotopy theory, one would expect crossed complexes to be useful in other areas of mathematics using cohomological techniques. Possibilities are algebraic geometry, differential topology (e.g. deformation theory) and in differential geometry. Is there a "crossed" version of differential forms? At present, crossed modules have found no applications in differential geometry.

(8.9) The classifying space BC of a crossed complex C generalises the classifying space BG of a groupoid G . But topological groupoids, and their classifying space, have also been found useful, for example in the theory of foliations [Law]. Are there similar uses of BC for C a topological crossed complex? Here again, it seems of interest to investigate simplicial T-objects in a category with finite limits.

(8.10) The notion of crossed complex has been given in a wide algebraic context [L]. Is there a corresponding simplicial T-complex notion in this general context? At the moment, T-complexes seem relevant only to group (or groupoid) cohomology.

(8.11) The definition of singular homology uses formal sums, and so leads to an abelian homology theory. The major methods of homological algebra, and their applications in diverse fields, are also abelian. Now Grothendieck has suggested (letters to L. Breen in 1975) that algebraic geometry requires a non-abelian cohomology, based on notions of n-groupoids which seem to be 'lax' or 'up-to-homotopy' versions of the n-groupoids of [B-H7]. (See also the remarks on 'n-catégories de Picard' on p.41 of [Br].) The bijections (7.7), (7.8) and the results of [B-H8] show that crossed complexes and simplicial T-complexes are relevant to non-abelian cohomology in the usual situations of algebraic topology. What relevance do they have to algebraic geometry?

(8.12) After the above was written, it turned out that the answer to (8.5) is: yes. (A cubical abelian group with connection defines an ω-groupoid in which

$$a \mathbin{+_i} b = a - \varepsilon_i \partial_i^1 a + b \ .)$$

This gives rise to cubical versions of Eilenberg-MacLane objects, something which was looked for in [Br].

(8.13) It would be interesting to have a notion of (simplicial) T-fibration generalising T-complexes in the same way that Kan fibrations generalise Kan complexes. The test of such a definition would be to obtain appropriate generalisations of Ashley's theorems. Such results could be useful in non-abelian versions of obstruction theory for the existence of sections. It might, though, be easier to think first of 'ω-groupoids over B' where B is a cubical complex with connection.

References

[Ad] J.F. Adams, *Infinite loop spaces*, Annals of Math. Studies No. 90, P.U.P. (1978).

[An] H. Ando, 'A note on Eilenberg-MacLane Invariant', Tohoku Math. J. (1957), 96-104.

[As] N.K. Ashley, *Crossed complexes and* T-*complexes*, Ph.D. thesis, University of Wales, (1978).

[Bℓ] A.L. Blakers, 'Some relations between homology and homotopy groups', Ann. of Math. 49 (1948), 428-461.

[B1] R. Brown, 'Cohomology with chains as coefficients', Proc.London Math. Soc., (3) 14 (1964), 545-565.

[B2] R. Brown, 'On Künneth suspensions', Proc. Camb. Phil. Soc., 60 (1964), 713-720.

[B3] R. Brown, 'Groupoids and van Kampen's theorem', Proc. London Math. Soc., (3) 17 (1967), 385-401.

[B4] R. Brown, *Elements of modern topology*, McGraw Hill, Maidenhead, (1968)

[B5] R. Brown, 'Higher dimensional group theory', in *Low Dimensional Topology, Vol.I*, Eds. R. Brown and T.L. Thickstun, London Math. Soc. Lecture Note Series No. 48, C.U.P. (1982).

[B6] R. Brown, 'On the homotopy classification of maps from a surface to the projective plane', (Preprint, Bangor, 1982).

[B-H 1] R. Brown and P.J. Higgins, 'On the second relative homotopy group of some related spaces', Proc. London Math. Soc., (3) (1978), 193-212.

[B-H 2] R. Brown and P.J. Higgins, 'Sur les complexes croisés, ω-groupoïds, et T-complexes', C.R. Acad. Sci. Paris, Sér. A 285 (1977), 997-999.

[B-H 3] R. Brown and P.J. Higgins, 'Sur les complexes croisés d'homotopie associés à quelques espaces filtrés', C.R. Acad. Sci. Paris, Sér. A 286 (1978), 91-93.

[B-H 4] R. Brown and P.J. Higgins, 'The algebra of cubes', J. Pure Appl. Alg. 21 (1981), 233-260.

[B-H 5] R. Brown and P.J. Higgins, 'Colimit theorems for relative homotopy groups', J. Pure Appl. Alg. 22 (1981), 11-41.

[B-H 6] R. Brown and P.J. Higgins, 'The equivalence of ω-groupoids and cubical T-complexes', Cah. Top. Géom. Diff. (3^e Coll. sur les catégories, dédié a Charles Ehresmann) 22 (1981), 349-370.

[B-H 7] R. Brown and P.J. Higgins, 'The equivalence of crossed complexes and ∞-groupoids', Cah. Top. Géom. Diff. (3^e Coll. sur les catégories dédié a Charles Ehresmann) 22 (1981), 370-386.

[B-H 8] R. Brown and P.J. Higgins, 'Crossed complexes and non-abelian extensions', *Int. Conf. on Category Theory, Gummersbach (1981)* Springer L.N.M. (to appear).

[B-H 9] R. Brown and P.J. Higgins, 'Crossed complexes and chain complexes with operators', (in preparation).

[B-S 1] R. Brown and C.B. Spencer, '$\mathcal{G}$-groupoids, crossed modules and the fundamental groupoid of a topological group', Proc. Kon. Ned. Akad. v. Wet. 79 (1976), 296-302.

[B-S 2] R. Brown and C.B. Spencer, 'Double groupoids and crossed modules', Cah. Top. Géom. Diff. 17 (1976), 343-362.

[C-G 1] A.M. Cegarra and A.R. Grandjean, 'Cohomología respecto a una variedad de una categoría', Sem. Mat. de Barcelona, Collect. Mat. 31 (1980), 3-26.

[C-G 2] A.M. Cegarra and A.R. Grandjean, 'Extensions and cohomology in categories of interest', (Preprint, Santiago de Compostela (1980).

[Cr] R.H. Crowell, 'On the van Kampen theorem', Pacific J. Math. 9 (1959), 43-50.

[C] E.B. Curtis, 'Simplicial homotopy theory', Adv. in Math. 6 (1971), 107-209.

[D1] M.K. Dakin, *Topological Groupoids, G-spaces and Hurewicz lifting functions*, M.Sc. thesis, University of Wales (1974).

[D2] M.K. Dakin, *Kan complexes and multiple groupoid structures*, Ph.D. thesis, University of Wales (1977).

[Do] A. Dold, 'Homology of symmetric products and other functors of complexes', Ann. of Math. 68 (1958), 54-80.

[Du] J. Duskin, *Simplicial methods and the interpretation of "triple" cohomology*, Mem. American Math. Soc. 3 (1975) No.163.

[Fe] H. Federer, *Lectures in algebraic topology*, Brown University, Providence, R.I. (1962).

[Ga] E.R.A. Garcia, *Cohomologia no abeliana en categorias de interes*, Thesis, University of Granada, (1981).

[Gℓ] P. Glen, *Realization of cohomology classes by torsors under hypergroupoids*, Ph.D. thesis, SUNY at Buffalo (1977).

[Go] D.H. Gottlieb, 'Covering transformations and universal fibrations', Ill. J. Math. 13 (1969), 432-437.

[Ha] V.L. Hansen, 'Spaces of maps into Eilenberg-MacLane spaces', Cam. J. Math. 33 (1981), 782-785.

[H1] P.J. Higgins, 'Presentations of groupoids, with application to groups', Proc. Camb. Phil. Soc. 60 (1964), 7-20.

[H2] P.J. Higgins, *Categories and groupoids*, van Nostrand, New York (1971).

[Hiℓ] R.O. Hill, Jr., 'A natural interpretation of the group cohomology group $H^n(Q, A)$, $n \geq 4$', Notices Amer. Math. Soc. 25 (1978), A - 351.

[Hin] S. Hintze, *Polysets, □-sets and cubical sets*. M.Phil. thesis, Warwick (1973).

[Hoℓ] D.F. Holt, 'An interpretation of the cohomology groups $H^n(G, M)$', J. Alg. 60 (1979), 307-318.

[How] J. Howie, 'Pullback functors and crossed complexes', Cah. Top. Géom. Diff. 20 (1979), 281-295.

[Hu] J. Huebschmann, 'Crossed N-fold extensions of groups and cohomology', Comm. Math. Helv. 55 (1980), 302-314.

[J] D.W. Jones, *Poly*-T-*complexes*, Ph.D. thesis, University of Wales (in preparation)

[K1] D.M. Kan, 'Abstract homotopy I', Proc. Nat. Acad. Sci. Washington 41 (1955), 1092-1096.

[K2] D.M. Kan, 'Abstract homotopy II', Proc. Nat. Acad. Sci. Washington 42 (1955), 225-228.

[K3] D.M. Kan, 'Abstract homotopy III', Proc. Nat. Acad. Sci. Washington 42 (1956), 419-421.

[K4] D.M. Kan, 'Abstract homotopy IV', Proc. Nat. Acad. Sci. Washington 42 (1956), 542-544.

[K5] D.M. Kan, 'On c.s.s. complexes', Amer. J. Math. 79 (1957), 449-476.

[K6] D.M. Kan, 'Functors involving c.s.s. complexes', Trans. Amer. Math. Soc. 87 (1958), 330-346.

[K7] D.M. Kan, 'A combinatorial definition of homotopy groups', Ann. of Math. 67 (1958), 282-312.

[La] K. Lamotke, *Semi-simpliziale algebraische Topologie*, Springer-Verlag, Berlin/New York (1968).

[Law] H.B. Lawson, Jr., 'Foliations', Bull. Amer. Math. Soc. 80 (1974), 369-418.

[L] A.S-T. Lue, 'Cohomology of groups relative to a variety', J. Algebra 69 (1981), 155-174.

[MacL] S. MacLane, 'Historical note', J. Algebra, 60 (1979), 319-320.

[Ma] J.P. May, *Simplicial objects in algebraic topology*, van Nostrand, Princeton (1968).

[Ma2] J.P. May, *The geometry of iterated loop spaces*, Springer Lecture Notes in Math. 271 (1972).

[M] J.C. Moore, 'Homotopie des complexes monoidaux I', Séminaire H. Cartan, Paris (1955).

[O] G. Orzech, 'Obstruction theory in algebraic categories I, II', J. Pure Appl. Alg. 2 (1972), 287-314, 315-340.

[S] G.B. Segal, 'Classifying spaces and spectral sequences', Publ. Math. I.H.E.S. 34 (1968), 105-112.

[vK] E.H. van Kampen, 'On the connection between the fundamental groups of some related spaces', Amer. J. Math. 55 (1933), 261-267.

[W] J.H.C. Whitehead, 'Combinatorial homotopy II', Bull. Amer. Math. Soc. 55 (1949), 453-496.

[G-Z] P. Gabriel and M. Zisman, *Calculus of fractions and homotopy theory*, Eng. d. Mat.u.i.Grenz 35 (Springer, Berlin, 1967).

[K-P-T] H. Kamps, D. Pümplun, M. Tholen (ed), *Proceedings International Conference on Category Theory, Gummersbach, July, 1981*, Springer Lecture Notes in Math. (to appear).

[Br] L. Breen, 'Extensions du groupe additif' Publ. Math. I.H.E.S. 48 (1978), 39-125.

[Du2] J. Duskin, 'Higher-dimensional torsors and the cohomology of topoi: the abelian theory', *Applications of Sheaves*, Springer Lecture Notes in Math. 753 (1979), 255-279.

[Gr] A. Grothendieck, *Catégories cofibrées additif et complexes cotangent relatif*, Springer Lecture Notes in Math. 79 (1968).